Introduction to Algebra Tutorial Week 3 Handout

02/10/17

1 Introduction

Please hand in the solutions for the following exercises next time: Q1, Q2, Q3, Q4(b)(c).

2 A practice Linear Diophantine Equation

We will have a look at the following Exercise today (as practice for Exercise 5):

Exercise. *Consider the linear Diophantine equation*

$$28x + 49y = 14$$

State whether or not this equation has an integer solution $(x, y) \in \mathbb{Z}^2$*. If not, state a reason for why. If so then find a closed form for all its integer solutions.*

Solution. Theorem 2.4.1 from lectures tells us that this equation has integer solutions if and only 14 is a multiple of $\gcd(28, 49)$. Staring at these numbers hard enough, we realise that $\gcd(28, 49) = 7$ so indeed this equation has integer solutions.

Theorem 2.4.5 tells us how we should find a closed form for these solutions. We must first apply the Euclidean algorithm forwards then backwards to 28 and 49 in order to find two integers u and v such that

$$28u + 49v = 7$$

So let's do that:

$$\begin{aligned} 49 &= 1 * 28 + 21 \\ 28 &= 1 * 21 + 7 \\ 21 &= 3 * 7 + 0 \end{aligned}$$

Now reversing the algorithm gives

$$\begin{aligned} 7 &= 28 - 1 * 21 \\ 7 &= 28 - 1 * (49 - 1 * 28) \\ 7 &= 28 - 1 * 49 + 1 * 28 \\ 7 &= 2 * 28 - 1 * 49 \end{aligned}$$

So $u = 2$ and $v = -1$. Hence the solutions to the equation are given by $(x_n, y_n)_{n \in \mathbb{Z}}$ where

$$x_n = \frac{14}{7}(2) + \frac{49}{7}n$$
$$x_n = 4 + 7n$$

$$y_n = \frac{14}{7}(-1) - \frac{28}{7}n$$
$$y_n = -2 - 4n$$

and we are done! (Make sure to check that these solutions indeed work!)