Numbers and Functions Tutorial Week 13 (apparently it could really be Week 8, who actually knows at this point tbh) Handout

13/11/17

1 A correction from last time

When doing Question 8 on Sheet 7 last time, I claimed that $\log 2 > 1$ as can be found in the solutions on KEATs. Someone asked me a question about why this is true and I said something nonsensical without realising that the statement $\log 2 > 1$ is incorrect (under the assumption that log is the logarithm with base e). The **correct** statement is that $\log 3 > 1$. The rest of the proof then holds fine once you replace every other occurence of 2 with 3.

2 Comments on Tutorial Answers (Sheet 7)

2.1 General Comments

- Be careful when you write out a statement that you intend to prove. Make sure to always write before it something like "We claim that". If you simply write a mathematical statement out then the implication is that statement is true.

2.2 Question 1

Throughout this question, it is important to make sure that the n_0 that you claim works is actually well-defined. By this I mean the following

1. You need to ensure that the algebraic equation defining your n_0 makes sense for all values of ε. Indeed, consider the following n_0 for Question 1.f):

$$n_0 = \left\lceil \sqrt{\frac{4}{\varepsilon^2} - 1} \right\rceil$$

   What happens if $4/\varepsilon^2 - 1 < 0$? Well, the square root is undefined for negative numbers so we certainly can't do that. Thankfully, in this question we can modify the algebraic manipulations to remove mention of the 1 and it all follows through (see the solutions on KEATs for more information)

2. You need to ensure that your n_0 is actually an element of $\mathbb{N}$. For example, consider the following n_0 for Question 1.c):

$$n_0 = \left\lceil \log_5\left(\frac{2}{\varepsilon}\right) \right\rceil + 1$$

   What happens when $2/\varepsilon < 1$? Well in this case, $\log_5(2/\varepsilon)$ is negative and so n_0 won't be an element of $\mathbb{N}$. We can remedy this by just taking

$$n_0 = \max\{1, \left\lceil \log_5\left(\frac{2}{\varepsilon}\right) \right\rceil + 1\}$$

   This works since if indeed $\log_5(2/\varepsilon)$ is negative then any n_0 works and we may as well just take 1 in that case.

2.3 Question 3,4,5,6,7

There were two main issues I saw with answers for these questions. The first is that some people were choosing an ε instead of fixing one. Some people managed to cleverly word their proofs to ensure that they were correct. However, the definition of convergence requires you to **fix** ε (i.e it cannot be chosen) and then exhibit a working n_0. I will write out the solution to Question 3 in more detail here so you can see what I mean.

Since $s_n \to l$, we know that

$$\forall \varepsilon > 0, \exists n_0(\varepsilon) \in \mathbb{N} \text{ such that } \forall n \geq n_0(\varepsilon) \text{ we have } |s_n - l| < \varepsilon \tag{1}$$

We want to prove that

$$\forall \varepsilon > 0, \exists m_0(\varepsilon) \in \mathbb{N} \text{ such that } \forall n \geq n_0(\varepsilon) \text{ we have } |t_n - 2l| < \varepsilon$$

To this end, fix $\varepsilon > 0$. By Statement 1, we know that there exists $n_0(\varepsilon/2)$ such that for all $n \geq n_0(\varepsilon/2)$ we have that $|s_n - l| < \varepsilon/2$. Then

$$|t_n - 2l| = |2s_n - 2l| = 2|s_n - l| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

We can thus take $m_0(\varepsilon) = n_0(\varepsilon/2)$.

The other issue I saw that people were not being very formal. Remember, every proof of convergence should pick out a specific $n_0(\varepsilon) \in \mathbb{N}$ to make sure that the sequence convergences.

Finally, another issue I saw was that some people used the Algebra of Limits to prove that the sequences converged. Most people did so successfully but the question stipulated that you can't use them ☺