Numbers and Functions Tutorial Week 4 Handout

16/10/17

1 Comments on Tutorial Answers

1.1 Question 5

b) English is unfortunately quite an ambiguous language - at least when it comes to writing logical statements. I saw many negations similar to

Alex will not ride or take a bus

This is certainly not wrong colloquially, its usually clear what this means. However, when it comes to writing down mathematical statements, 'usually' is not really good enough. Watch what happens when I put brackets around different parts of the above negation:

Alex will not (ride or take a bus)

Alex will (not ride) or (take a bus)

If you think about it, the second statement is a perfectly reasonable interpretation of the statement. This is because English is in some way flexible as to what words the 'not' prefix actually affects. It is therefore much better if you make sure to negate both parts:

Alex will neither ride nor take a bus

c) Note that in $0 < x \leq 1$, there is an implicit 'AND':

$$0 < x \text{ AND } x \leq 1$$

so the negation is

$$0 \geq x \text{ OR } x > 1$$

Also, some people gave the answer in terms of interval as such:

$$x \in (-\infty, 0] \cup (1, \infty)$$

which is a good answer but only if $x \in \mathbb{R}$! Remember, intervals are special notation for real numbers and we didn't specify beforehand that $x \in \mathbb{R}$. All we know that x is a member of an **ordered set**[1] containing 0 and 1.

- The 'NOT' keyword is used to negate statemnts, not to take the complement of a set. In the case of statement C, the correct answer is $\mathbb{R}\backslash A$ and not 'NOT A'. A^c is also acceptable - it is read as "complement of A". It is usually used when the **ambient set** that A is contained in is fixed and known. In this case, the ambient set is $\mathbb{R}$.

2 Comments on Lecture Notes

2.1 Proofs

2.1.1 The necessity of proofs in mathematics

Here is something I would add to this section. It is often thought that the need for rigour and proof in mathematics is just an excuse for mathematicians to stroke their egos and insist on annoying specificity. This can certainly seem the case from the example in this section on Fermat numbers. *Who cares* if F_5 is

[1] There are various kinds of ordered sets but what's important to understand here is that we have a symbol $<$ (together with its brethren $\leq, >, \geq$) which give a so-called **ordering relation** on our set.

composite and not prime? Well it turns out that's actually quite important because knowing what numbers are prime is used directly in keeping your credit card details safe when making an online transaction, for example.

Here is another example which, in some ways, is even more important than the previous. Consider how you came into uni today. Maybe it was the tube or the bus or even possibly by bike. Why do you trust such modes of transportation to keep you safe? The person who built such a vehicle was likely an engineer who consulted some physicists (or just a physics book). These physicists guaranteed the engineer that his or her vehicle will obey certain laws of physics which can be predicted by certain equations. Let's say one such law is Newton's Second Law of Motion: force = mass × acceleration. Phrased differently, Newton's Second Law is

$$F = m\frac{d^2x}{dt^2}$$

which is just a second order differential equation. Now, these physicists likely consulted some mathematicians (or just a mathematics book) to know the solutions to these equations and to see the *proof* for why these solutions are always of a certain form. The physicists can then be satisfied that there will be no surprise solutions when applying this equation to their physics. The engineers can then be satisfied that their vehicle won't crash and burn in the hands of customers. You yourself can then be satisfied that your vehicle will keep you safe.

Note that none of this implies that the mathematical proof is more important than any of the physics or engineering or any other application of the mathematics. Mathematical proof just forms one of the many fundamental pillars of such things.

2.1.2 An example proof

Here is one of the first proofs that a mathematician might encounter:

Theorem. *$\sqrt{2}$ is not a rational number.*

Proof. We shall prove this theorem by contradiction. That is to say, we shall make an assumption which will lead us (through a series of correct deductions) to something obviously false. We can then deduce that our original assumption **must** be incorrect.

So suppose, for a contradiction, that $\sqrt{2}$ is rational. This means that there exist $a, b \in \mathbb{Z}$ such that

$$\sqrt{2} = \frac{a}{b}$$

The first thing we can do is tackle the square root. Let's square both sides and simplify to get

$$\begin{aligned} 2 &= \frac{a^2}{b^2} \\ 2b^2 &= a^2 \qquad (1) \end{aligned}$$

This means that a^2 is an even number (it is a multiple of two). But if the square of a number is even then the number itself is also even so a is even. That means we can write $a = 2k$ for some other $k \in \mathbb{Z}$. Plugging this into Equation 1 gives $2b^2 = (2k)^2 = 4k^2$. We can cancel a 2 here to give $b^2 = 2k^2$. But then b^2 is even. Following the same logic as for a^2, we then have that $b = 2m$ for some $m \in \mathbb{Z}$. We have therefore proven that, if $\sqrt{2}$ is indeed even then a and b must have a factor of 2 in common. We can cancel the factor of 2 to get that

$$\sqrt{2} = \frac{k}{m}$$

But then we can repeat this whole process again to knock off another factor of 2 off both the numerator and denominator! No matter how many times we knock off a factor of 2, we can always prove there will be another factor of 2 that we can knock off. This is a contradiction as an integer cannot be cleanly divided out by 2 infinitely many times. So our original assumption that $\sqrt{2}$ is rational must be an incorrect assumption so we conclude that $\sqrt{2}$ is irrational. □