Numbers and Functions Tutorial Week 3 Handout

09/10/17

1 Comments on Tutorial Answers

1.1 General

- Don't be afraid to leave your rough work in your answers - it can help me in identifying errors.
- If you want to add something to your answers from what I've said in class, please use a different colour pen! Otherwise I end up marking myself and becoming my own teacher!
- When possible, please write on a separate sheet of paper.
- Try and get used to writing sets symbolically using set-builder notation or intervals (for real numbers). I saw many answers similar to '$x \geq 3$'. What is x here? A real number? Integer? It is not immediately clear what this expression refers to. The correct way to write it (in the case that it refers to real numbers) would be either $[3, \infty)$ or

$$\{ x \in \mathbb{R} \mid 3 \leq x < \infty \}$$

1.2 Question 1

Watch out for $\mathbb{N}$ vs $\mathbb{Z}$. The former consists of the positive integers (and usually 0 depending on convention). The latter is *all* integers.

1.3 Question 2

$K \backslash (M \cup N)$ consists of all those King's students who are not male **and** not taking Numbers and Functions. What is useful here is **De Morgan's Law**:

$$K \backslash (M \cup N) = (K \backslash M) \cap (K \backslash N)$$

1.4 Question 3

- The interval notation is used only for real numbers. All the sets involved are sets containing real numbers.
- Don't forget to write single element sets and sets with explicit elements with brackets: $\{ 0 \}$ or $\{ -1, 1 \}$.

1.5 Question 4

b) $A \subset B$ means "either A is a **proper** subset of B or $A = B$". See also last week's handout regarding the usage of $\subset$ vs $\subseteq$.

c) An interesting point was raised about the case when $A \cap B = \varnothing$. In this case, is $A \cap B \subset A \cup B$? The empty set is certainly a subset of $A \cup B$. This is called **vacuous truth**. The definition of a subset requires that all elements of $A \cap B$ be elements of $A \cup B$. If $\varnothing$ were not a subset of $A \cup B$ then there would exist an element of $\varnothing$ that is not an element of $A \cup B$. But $\varnothing$ *has no elements* so this cannot possibly be true. There are no elements to act as counter-examples! We conclude, therefore, that $\varnothing$ is indeed a subset of $A \cup B$.

d) Let $A = B$ where A is any set you like. Then $A \backslash B = A \backslash A = \varnothing$ and $B \backslash A = A \backslash A = \varnothing$.

e) Consider $A = \mathbb{N}$ and $B = \varnothing$. Then $A \backslash B = \mathbb{N}$ but $B \backslash A = \varnothing$.

f) Let $B = A$. Then $A \cup B = A \cup A = A$ and $A \cap B = A \cap A = A$.

2 Comments on Lecture Notes

2.1 Functions

2.1.1 A Theorem from last time

Last time I mentioned the following theorem; here's a proof for it. You will soon be introduced to proofs in this course - consider this a taster!

Theorem. *Let A and B be sets and $f : A \to B$ a function. Then f has an inverse if and only if f is bijective.*

Proof. We are being asked to prove an if and only if statement. This means that we have two proofs to do: the forwards and backwards directions. Let's start with the forward direction.

The forward direction reads as follows: If f has an inverse then f is bijective. So let's start off by supposing that we are given a function $f : A \to B$ that has an inverse $f^{-1} : B \to A$. We need to show that f is both injective and surjective.

Let's prove injectivity first. We need to show that $f(x) = f(y)$ implies that $x = y$. Post-composing with f^{-1} (whose existence we *assumed*), we get $(f^{-1} \circ f)(x) = (f^{-1} \circ f)(y)$. But by the definition of an inverse, we know that $f^{-1} \circ f$ is simply the identity and so these cancel out to leave us with the equation $x = y$. So f is injective.

Now let's show that f is surjective. We need to show that for all $y \in B$ there exists an $a \in A$ such that $f(a) = b$. Well just post-compose by f^{-1} : we get $(f^{-1} \circ f)(a) = f^{-1}(b)$. Once again, $f^{-1} \circ f$ cancels to leave us with $a = f^{-1}(b)$. Hence we have found an a that maps to b under f - so f is surjective.

This completes the proof of the forward direction: if f has an inverse then f is bijective. We must now prove the backwards direction.

The backwards direction reads as follows: If f is bijective then f has an inverse. We shall explicitly construct the inverse $g : B \to A$ of f in the following way. Given $b \in B$, let $g(b) = a$ where a is the unique element of A such that $f(a) = b$. We know that such an a exists since f is surjective. We know that such an a is unique since f is injective. Hence g is a well-defined function: every element of B maps to some element of A under g (by surjectivity of f) and no element of B is mapped to multiple elements of A (by injectivity of f). It remains to check that g is an inverse of f. Let $a \in A$. Then $f(a) = b$ for some $b \in B$. Then $g(f(a)) = g(b) = a$ so g is a **left-inverse**. Similarly, $f(g(b)) = f(a) = b$ and so g is a **right-inverse**. Hence g is an inverse for f. □

As an aside to the above Theorem, we also have the following:

Proposition. *Let A and B be sets and $f : A \to B$ a function. If f has an inverse then its inverse is unique.*

Proof. Suppose that g and h are two inverses to f. We claim that, in fact, $g = h$. Denote by id_A the identity on A and, similarly, id_B. By the associativity of composition, we have

$$g = g \circ \mathrm{id}_B = g \circ (f \circ h) = (g \circ f) \circ h = \mathrm{id}_A \circ h = h$$

as claimed. □

3 Logic

3.1 Negating Propositions and Contrapositives

Why should we care about negating propositions? That is certainly a good question and it is one that is often not particularly well answered. With every correctly formulated proposition we associate a **truth value**: either **true** or **false**. The act of proving a statement is the act of associating (sometimes gruelingly!) a truth value to a statement[1]. Suppose someone outrageously claims the false statement

The sky is green

How do you prove them wrong? Naturally your first instinct is to say

No, **the sky is not green**

[1]For the interested: it is not always possible to prove something is either true or false - see the 'Continuum Hypothesis' and Gödel's incompleteness theorems

The bolded statement here is the negation of the original statement - it takes the opposite truth value (and in this case, it *is* the truth). The moral here is: *the first step to showing a false statement is false is to take its negation and argue that the new statement is true.*

The negation also crops up in less obvious ways. Let's say we have two statements A and B and we want to show that in fact $A \implies B$ (in other words, the truth of A implies the truth of B). Sometimes this is not an easy task. Now, the contrapositive of $A \implies B$ which is

$$(\text{NOT } B) \implies (\text{NOT } A)$$

actually has the same truth value as $A \implies B$. It happens fairly often in mathematics that the contrapositive is much easier to prove than the direct statement. This could be because framing the statements in terms of their negations unlock new insights that we previously did not realise. Therefore, if we can establish the truth value of the contrapositive, we can establish the truth value of the original statement!

It is important to note at this stage: **do not mix up the converse with the contrapositive!!!!** The converse of an implication $A \implies B$ is $B \implies A$. The truth of the converse is almost never related to the truth of the original statement. There is an example in your notes but here is another.

$$\begin{aligned} \text{Original: } & A \text{ is a cat} \implies A \text{ is an animal} \\ \text{Converse: } & A \text{ is an animal} \implies A \text{ is a cat} \\ \text{Contrapositive: } & A \text{ is not an animal} \implies A \text{ is not a cat} \end{aligned}$$