Topology - 6CCM327A

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Chapter 1

Topological Spaces and Continuous Functions

1.1 Topological Spaces

Definition 1.1.1. *A is open if $\forall\, x \in A, \exists\, \varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq A$*

Proposition 1.1.2. *Let $\tau = \{A \subseteq \mathbb{R} \,|\, A \text{ is open}\}$. Then τ satisfies the following properties:*

1. *$\varnothing, \mathbb{R} \in \tau$*
2. *τ is closed to arbitrary unions*

   $\forall\, I, \forall\, \{U_i\}_{i \in I} \subseteq \tau,\ (\bigcup_{i \in I} U_i) \in \tau$
3. *τ is closed to taking finite intersections*

   $\forall\, U_1, U_2, \ldots, U_n \in \tau$ *(U_i open)*, $(\bigcap_{i=1}^{n} U_i) \in \tau$ *is open*

Proof. We prove parts 2 and 3:

Part 2: Let $U = \bigcup_{i \in I} U_i,\ x \in U$

$$x \in U \implies \exists\, i \in I \text{ such that } x \in U_i$$

$\exists\, \varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq U_i \subseteq U$

Since $x \in U$ is arbitrary $\implies$ U is open

Part 3: Let $U = \bigcap_{i=1}^{n} U_i$, $x \in U$

$\forall\, i \leq n, x \in U_i$

$\forall\, i, \exists\, \varepsilon_i > 0$ such that $(x - \varepsilon_i, x + \varepsilon_i) \subseteq U_i$

Take $\varepsilon = min_{i \leq n}\{\varepsilon_i\} > 0$. Then $\forall\, i \leq n, (x - \varepsilon, x + \varepsilon)$
$\subseteq (x - \varepsilon_i, x + \varepsilon_i) \subseteq U_i$

$\implies (x - \varepsilon, x + \varepsilon) \subseteq U \implies$ U is open

□

Definition 1.1.3. *Let X be any set (finite or infinite). A topology τ on X is a collection of subsets of X ($\tau \subseteq 2^X$) satisfying the following axioms:*

- $\varnothing$, $\mathbb{R} \in \tau$
- $\forall\, I, \forall\, \{U_i\}_{i \in I} \subseteq \tau$, $\bigcup_{i \in I} U_i \in \tau$
- $\forall\, n, \forall\, U_1, \ldots, U_n \in \tau$, $\bigcap_{i=1}^{n} U_i \in \tau$

Remark.

- *Each element of τ is a subset of X*
- *We say that a set $U \in \tau$ is open (relatively to τ)*
- *A closed set is the complement of an open set i.e $C = X \backslash U$, $U \in \tau$*

Example 1.1.4. *Consider $X = \{a, b, c\}$.*

A topology on X is $\tau = \{\varnothing, X, \{a, b\}\}$. The closed sets are $\varnothing, X, \{c\}$.

Another topology on X is $\tau = \{\varnothing, X, \{a, b\}, \{c\}\}$. The closed sets are $\varnothing, X, \{a, b\}, \{c\}$.

$\tau = \{\varnothing, X, \{a, b\}, \{b\}, \{c\}\}$ is not a topology as it violates the second axiom.

Example 1.1.5. *Consider $X = \mathbb{R}$ and $\tau = \{$open in the usual sense$\}$ then τ is called the* ***standard topology on $\mathbb{R}$****.*

Example 1.1.6. *Let X be an arbitrary set. Then we can define the following two topologies on X:*

- $\tau_1 = \{\varnothing, X\}$ - ***trivial topology***
- $\tau_2 = 2^X = \{A \,|\, A \subseteq X\}$ - ***discrete topology***

Example 1.1.7. *Consider $X = \mathbb{R}$ and $\tau = \{(a,b) \,|\, a < b\}$. Then τ satisfies the first and third axioms but violates the second. Indeed, consider $U_1 = (1,2), U_2 = (3,4)$. $U_1, U_2 \in \tau$ but $U_1 \cup U_2 \notin \tau$.*

Proposition 1.1.8. *Let X be an arbitrary set. Consider $\tau_f = \{A \subseteq X \,|\, X \backslash A$ is finite$\} \cup \{\varnothing\}$. Then τ is a topology on X and is called the* ***finite complement topology on X****.*

Proof. $\varnothing, X \in \tau_f$ is trivial.

Let $\{U_i\}_{i\in I} \subseteq \tau_f$
If $\forall\, i, U_i = \varnothing$ then $\bigcup_{i\in I} U_i = \varnothing \in \tau_f$
Otherwise, $X \backslash U = X \backslash (\bigcup_{i\in I} U_i) = \bigcap_{i\in I}(X \backslash U_i) \subseteq X \backslash U_i \implies X \backslash U$ is finite $\implies U \in \tau_f$

Now consider $U_1, \ldots, U_n \in \tau_f$
$X \backslash U = \bigcup_{i=1}^{n}(X \backslash U_i)$ is a finite union of finite sets $\implies X \backslash U$ is finite $\implies U \in \tau_f$ □

Definition 1.1.9. *Let X be an arbitrary set and τ, τ' two topologies defined on X. We say that τ' is* ***finer*** *than τ if $\tau \subseteq \tau'$ (τ is* ***coarser*** *than τ').*
If either τ is finer or coarser than τ' then we say that τ, τ' are ***comparable****.*

Proposition 1.1.10. *Consider $X = \mathbb{R}$. Let τ be the standard topology on $\mathbb{R}$ and τ_f the finite complement topology on $\mathbb{R}$. Then τ is finer than τ_f.*

Proof. We need to show that $\tau_f \subseteq \tau$.

$\tau_f = \{U \subseteq \mathbb{R} \,|\, \mathbb{R} \backslash U = \{\text{finite or } \mathbb{R}\}\}$
Let $U \in \tau_f$ then $U = \{\varnothing \text{ or } \mathbb{R} \backslash \{x_1, \ldots, x_n\}\}$
If $U = \varnothing$ then $U \in \tau$ so suppose $U \neq \varnothing$

Then $\mathbb{R}\backslash\{x_1, \ldots, x_n\} \in \tau$ as we can always find points with a small enough neighbourhood in $\mathbb{R}$ such that the points are open.
$\implies \tau_f \subseteq \tau$
In fact, $\tau_f \subsetneq \tau$ (τ is strictly finer than τ_f e.g $(0, 1) \in \tau\backslash\tau_f$) □

1.2 Topology Bases

Definition 1.2.1. *Let X be a set. A* ***topology basis on X*** *is a collection $\mathcal{B}$ of subsets of X satisfying:*

1. $\forall\, x \in X, \exists\, B \in \mathcal{B}$ *such that* $x \in B$
2. $\forall\, B_1, B_2 \in \mathcal{B}, \forall\, x \in (B_1 \cap B_2), \exists B_3 \in \mathcal{B} \,|\, x \in B_3 \subseteq (B_1 \cap B_2)$

(The finite intersection property $B_1, B_2 \in \mathcal{B} \implies B_1 \cap B_2 \in \mathcal{B}$ is stronger than the second axiom.)

Proposition 1.2.2. *Let $\mathcal{B}$ be a topology basis. Then it generates the following topology $\tau = \tau_B$ on X: For $U \subseteq X, U \in \tau$ if*

$$\forall\, x \in U, \exists\, B \in \mathcal{B} \text{ such that } x \in B \subseteq U$$

Proof.

1. We first show that $\varnothing, X \in \tau$

   $\varnothing$ - satisfied trivially
   X - $\forall\, x \in X, \exists\, B \in \mathcal{B}$ such that $x \in B \subseteq X \implies X \in \tau$

2. Now we show that for $\{U_i\}_{i\in I} \subseteq \mathcal{B}$ then $U = \bigcup_{i\in I} U_i \in \tau$

   If $x \in U, \exists\, i_0$ such that $x \in U_{i_0}, U_{i_0} open$
   $\implies \exists B \in \mathcal{B}$ such that $x \in B \subseteq U_{i_0} \subseteq U$

3. Finally we prove by induction that $\forall\, U_1, \ldots, U_n \in \tau,\ \bigcap_{i=1}^{n} \in \tau$

   By induction, we may assume that n=2.
   Consider $U_1, U_2 \in \tau$. We must show that $U = U_1 \cap U_2 \in \tau$. Let $x \in U$. Since U_1 is open, we can excise an open neighbourhood around x in U_1. The same can be said for U_2.

Since $x \in U_1, U_2 \in \tau, \exists\, B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$. By the same argument on $U_2 \in \tau, \exists\, B_2 \in \mathcal{B} \subseteq U_2 \implies x \in B_1 \cap B_2 \subseteq U_1 \cap U_2 = U$. By the second basis axiom, we can choose a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$. Then $x \in B_3$ and $B_3 \subseteq U_1 \cap U_2$, so $U_1 \cap U_2$ belongs to τ.

We now assume that the fact is true for $n-1$ and prove it for n. Now:

$$U_1 \cap \cdots \cap U_n = (U_1 \cap \cdots \cap U_{n-1}) \cap U_n$$

By hypothesis, $U_1 \cap \cdots \cap U_{n-1}$ belongs to τ. By the result just proved, the intersection of $U_1 \cap \cdots \cap U_{n-1}$ and U_n also belongs to τ.

□

Remark. *τ is the coarsest topology that makes all sets $B \in \mathcal{B}$ open.*

Example 1.2.3. *Let $X = \mathbb{R}, \mathcal{B} = \{(a,b) \,|\, a < b, a, b \in \mathbb{R}\}$*

1. *Let $\varepsilon = 1$ then $\forall\, x, x \in (x-\varepsilon, x+\varepsilon)$ - first axiom satisfied*

2. *Consider $(a,b), (c,d) \in \mathcal{B}$ then $(a,b) \cap (c,d) = \{\varnothing$ or $(\alpha, \beta)\}$ for some $\alpha, \beta \in \mathbb{R}$ - second axion satisfied*

This basis generates the standard topology on $\mathbb{R}$.

Example 1.2.4. *Let $X = \mathbb{R}^2, \mathcal{B} = \{B_r(x) \,|\, x \in \mathbb{R}^2, r > 0\}$*

1. *Let $r = 1$ then $B_r(x) \in \mathcal{B}$ - first axiom satisified*

2. *The finite intersection property is not satisfied but we can always excise a small open neighbourhood around a point in the intersection of two open balls - second axion satisfied*

This basis generates the standard topology on $\mathbb{R}$.

Example 1.2.5. *Let $X = \mathbb{R}^2, \mathcal{B} = \{$open rectangles$\} = \{(a,b) \,\mathrm{x}\, (c,d) \,|\, a < b, c < d\}$*

1. *Consider $(x,y) \in \mathbb{R}^2$ and take $B = (x-1, x+1)\mathrm{x}(y-2, y+2)$ then $B \in \mathcal{B}$ - first axiom satisfied*

2. $U_1 \cap U_2 = \{\varnothing$ *or open rectangle*$\}$ *- finite intersection property satisfied*

This basis defines the same standard topology on $\mathbb{R}$ *as the previous example.*

Example 1.2.6. *Let X be an arbitrary set,* $\mathcal{B} = \{B_x = \{x\} \mid x \in X\}$

1. $\forall\, x \in X, x \in B_x$ *- first axiom satisfied*
2. $B_x \cap B_y = \begin{cases} \{x\} & \text{if } x = y \\ \varnothing & \text{if } x \neq y \end{cases}$ *- finite intersection property satisfied*

Proposition 1.2.7. *Let X and $\mathcal{B}$ be as in the previous example. Then the topology by $\mathcal{B}$ on X is $\tau_B = 2^X$ (discrete topology).*

Proof. Consider $U \subseteq X$. $\forall\, x \in U, x \in B_x \subseteq U$. Since we have $B_x \in \mathcal{B}$, $\mathcal{B}$ must generate the discrete topology. □

Lemma 1.2.8. *Let $\mathcal{B}$ be a topology basis on a set X and let τ be the topology generated by $\mathcal{B}$. Then:*

$$\tau = \left\{ \bigcup_{i \in I} B_i \mid B_i \in \mathcal{B} \right\}$$

In other words, τ consists of arbitrary unions of basic neighbourhoods.

Proof. Let $\tau' = \left\{ \bigcup_{i \in I} B_i \mid \forall\, i \in I, B_i \in \mathcal{B} \right\}$

We must prove that $\tau = \tau'$.

$\underline{\tau' \subseteq \tau}$: $\forall\, i, B_i \in \tau \implies \bigcup_{i \in I} B_i \in \tau$ by the second axiom of a topology.

$\underline{\tau \subseteq \tau'}$: Suppose that $U \in \tau$. Then $\forall\, x \in U, \exists\, B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$.
$\implies U = \bigcup_{x \in U} B_x$.
$\implies U \in \tau$.

□

Lemma 1.2.9. *Let (X, τ) be a topological space and $\mathcal{C}$ a collection of open sets in X. If $\forall\, U \in \tau, \forall\, x \in U, \exists\, C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a topology basis that generates τ.*

- *$\mathcal{C}$ is not assumed to be a topology basis.*
- *All $C \in \mathcal{C}$ are assumed to be open.*

Proof. First, we show that $\mathcal{C}$ is a topology basis.

1. Let $x \in X$, choose $U = X$. By assumption, $\exists\, C \in \mathcal{C}$ such that $x \in C \subseteq U = X$.

2. Consider $C_1, C_2 \in \mathcal{C}, U = C_1 \cap C_2 \in \tau$ is open.
   Let $x \in U$. By assumption, $\exists\, C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq U = C_1 \cap C_2$.

Now, let τ' be the topology generated by $\mathcal{C}$. We claim that $\tau = \tau'$.

$\underline{\tau \subseteq \tau'}$: Let $U \in \tau$. To check that $U \in \tau'$, we have to show that $\forall\, x \in U, \exists\, C \in \mathcal{C}$ such that $x \in C \subseteq U$ which is true by the condition of the lemma.

$\underline{\tau' \subseteq \tau}$: Let $U \in \tau'$. By Lemma 1.2.8, we may write $U = \bigcup_{i \in I} C_i$ for some $C_i \in \mathcal{C} \implies$ U is open as a union of open sets.

□

Example 1.2.10. *Consider $X = \mathbb{R}$ equipped with the standard topology and $\mathcal{C} = \{(a,b) \,|\, -\infty < a < b < \infty\}$.*
We can see that this is a basis for the standard topology by taking $x \in (a,b) \subseteq \mathbb{R}$. Now consider $C = (x - \varepsilon, x + \varepsilon) \subseteq (a,b)$ for sufficiently small ε. Indeed, $C \in \mathcal{C}$ so by Lemma 2, $\mathcal{C}$ is a basis for $\mathbb{R}$ equipped with the standard topology.

Example 1.2.11. *Consider $X = \mathbb{R}^2$ equipped with the standard topology and $\mathcal{B} = \{B_r(x) \,|\, x \in \mathbb{R}^2, r > 0\}$.*
Indeed, $\mathcal{B}$ generates the standard topology on $\mathbb{R}^2$ as given any $U \subseteq \mathbb{R}^2$ open and any $x \in U$, we can always a squeeze an open ball of sufficiently small radius r around x.
Now consider $\mathcal{B}' = \{(a,b)\text{x}(c,d) \,|\, -\infty < a < b < \infty, -\infty < c < d < \infty\}$. This set is also a topology basis for $\mathbb{R}^2$ equipped with the standard topology.

Lemma 1.2.12. *Let $\mathcal{B}, \mathcal{B}'$ be topology bases and τ, τ' the corresponding topologies. The following are equivalent:*

1. *τ' is finer than τ*

2. *$\forall\, B \in \mathcal{B}, \forall\, x \in B, \exists\, B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$*

Remark. *For topological equality ($\tau = \tau'$), we need to check the condition both ways.*

Proof.

$\Longrightarrow$: Assume $\tau \subseteq \tau'$. Let $B \in \mathcal{B}, x \in B. B \in \tau \subseteq \tau' \implies x \in B \in \tau$. Since τ' is generated by $\mathcal{B}', \exists\, B' \in \mathcal{B}', x \in B' \subseteq B$.

$\Longleftarrow$: Assume 2. and let $U \in \tau. \forall\, x \in U, \exists\, B \in \mathcal{B}, x \in B \subseteq U$.
By 2. $\exists\, B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
$\implies U \in \tau'$

□

1.3 Product topology on $X \times Y$

Proposition 1.3.1. *Let X, Y be two topological spaces. The product topology on $X \times Y$ is the topology generated by:*

$$\mathcal{B} = \{U \times V \mid U \textit{ is open in } X,\ V \textit{ is open in } Y\}$$

Proof.

1. X open in X, Y open in Y $\implies X \times Y \in \mathcal{B}$
2. Consider $U \times V, U' \times V' \in B$ - first axiom satisfied
   $(U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V') \in \mathcal{B}$ - finite intersection property satisfied

□

Example 1.3.2. *Consider $X = Y = \mathbb{R}$ both equipped with the standard topology and $X \times Y = \mathbb{R}^2$.*
The product topology is generated by $\{I \times J \mid I, J \text{ open in } \mathbb{R}\}$ and is equivalent to the standard topology on $\mathcal{R}^2$.

Theorem 1.3.3. *Let $\mathcal{B}$ be a topology basis on X and $\mathcal{C}$ a topology basis on Y. Then the collection*

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

Proof. We shall prove this theorem by invoking Lemma 2. We must show that $\mathcal{D}$ consists of open sets of $X \times Y$ and that $\forall$ open $W \subseteq X \times Y, \forall\, (x, y) \in W, \exists\, D \in \mathcal{D}$ such that $(x, y) \in D \subseteq W$.

1. We have that $\forall B \in \mathcal{B}$, B is open in X and $\forall, C \in \mathcal{C}$, C is open in Y by the definition of a topology basis. Hence by the definition of the product topology, we have that $B \times C$ is open in the product topology. Thus $\mathcal{D}$ consists of open sets of $X \times Y$.

2. Consider $W \subseteq X \times Y$ open and $(x, y) \in W$. By the definition of the product topology, $\exists U \subseteq X, V \subseteq Y$ such that $(x, y) \in U \times V \subseteq W$. Since $\mathcal{B}, \mathcal{C}$ are topology bases for X and Y respectively, we have that $\exists B \in \mathcal{B}, x \in B \subseteq U$ and $\exists C \in \mathcal{C}, y \in C \subseteq V$.
   Now, $(x, y) \in B \times C \subseteq U \times V \subseteq W$ and hence by Lemma 2, $\mathcal{D}$ is a topology basis for the product topology on $X \times Y$.

□

1.4 Subspace topology

Definition 1.4.1. *Let X be a space endowed with a topology τ. If Y is a subset of X, the collection*

$$\tau_Y = \{Y \cap U \,|\, U \in \tau\}$$

*is a topology on Y, called the **subspace topology**. With this topology, Y is called a **subspace** of X; its open sets consist of all intersections of open sets of X with Y.*

Lemma 1.4.2. *Let (X, τ) be a topological space and $Y \subseteq X$ endowed with the subspace topology τ_Y. If $A \in \tau_Y$ and $Y \in \tau$ then $A \in \tau$.*

Proof. If $A \in \tau_Y$, we have that $A = Y \cap U$ for some $U \in \tau$. Since $Y \in \tau$, we see that $A = Y \cap U \in \tau$ since the intersection of two open sets is again open. □

Lemma 1.4.3. *If $\mathcal{B}$ is a basis for a topology on X and $Y \subseteq X$ equipped with the subspace topology τ_Y then the following collection*

$$\mathcal{B}_Y = \{B \cap Y \,|\, B \in \mathcal{B}\}$$

is a topology basis for τ_Y.

Proof.

1. We show that $\mathcal{B}_Y$ consists of open sets of Y.
   Since $\mathcal{B}$ is a basis for the topology on X, we have that $\forall B \in \mathcal{B}$, B is open in X. Now, by the definition of the subspace topology, $B \cap Y$ is open in Y. Thus $\mathcal{B}_Y$ consists of open sets of Y.

2. We now show that $\forall\, B_1, B_2 \in \mathcal{B}_Y, \forall\, x \in B_1 \cap B_2, \exists\, B_3 \in \mathcal{B}_Y$ such that $x \in B_3 \subseteq B_1 \cap B_2$.
   Consider $B_1, B_2 \in \mathcal{B}_Y$ and $x \in B_1 \cap B_2$. We have that $B_1 = U_1 \cap Y$ and $B_2 = U_2 \cap Y$ for some $U_1, U_2 \subseteq X$ open. Since $\mathcal{B}$ is a topology basis for X, we can find basis elements $A_1 \subseteq U_1, A_2 \subseteq U_2$ such that $x \in A_1 \cap Y, A_2 \cap Y$. Hence we have that $x \in B_3 = (A_1 \cap Y) \cap (A_2 \cap Y)$.

□

Example 1.4.4. *Consider* $Y = [-1, 1] \subseteq \mathbb{R}$ *and* $\mathcal{B} = \{(a, b) \,|\, -\infty < a < b < \infty\}$ *a topology basis on* $\mathbb{R}$.
By Lemma 1.4.3, the following is a topology basis for Y:

$$\mathcal{B}_Y = \{(a, b) \,|\, -1 < a < b < 1\} \cup \{(a, 1] \,|\, -1 \leq a < 1\} \cup \{[-1, b) \,|\, -1 < b \leq 1\}$$

Theorem 1.4.5. *Let X and Y be topological spaces,* $A \subseteq X, B \subseteq Y$ *subspaces. Consider* $\tau_{A\times B}$ *the product topology on* $A \times B$ *and* $\tilde{\tau}_{A\times B}$ *the subspace topology on* $A \times B \subseteq X \times Y$. *Then* $\tau_{A\times B} = \tilde{\tau}_{A\times B}$.

Proof. Consider the set $U \times V$, a general basis element for $X \times Y$ where U is open in X and V is open in Y. We can see that $(U \times V) \cap (A \times B)$ is the general basis element for the subspace toplogy on $A \times B$. Now

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$$

Since $U \times A$ and $V \times B$ are the general open sets for the subspace topologies on A and B respectively, the set $(U \cap A) \times (V \cap B)$ is the general basis element for the product toplogy on $A \times B$. Since the two bases are equivalent, the two topologies must be equivalent. □

1.5 Closed sets and limits points

Definition 1.5.1. *A subset A of a topological space X is* **closed** *if* $X \backslash A$ *is open. A is* **clopen** *if it is both open and closed.*

Example 1.5.2. *Consider* $X = (0,1) \cup (2,3) \subseteq \mathbb{R}$. $A = (0,1)$ *is a clopen subset of X.*

Remark. *There are no non-trivial (i.e not* $\varnothing$ *or* $\mathbb{R}$*) clopen subsets of* $\mathbb{R}$*. (equivalent to the notion of* $\mathbb{R}$ *being* **connected***).*

Theorem 1.5.3. *Let X be a topological space. Then the following conditions hold:*

1. $\varnothing$ *and X are closed*
2. $\forall \{C_i\}_{i \in I}, C_i$ *closed* $\implies \bigcap_{i \in I} C_i$ *is closed*
3. $\forall\, i = 1 \ldots n, C_i$ *closed* $\implies \bigcup_{i=1}^{n}$ *is closed*

Proof.

Part 1: $\varnothing$ and X are closed because they are the complements of the open sets X and $\varnothing$ respectively.

Part 2: Let $\{C_i\}_{i \in I}$ be a collection of closed sets in X. It suffices to show that $X \backslash (\bigcap_{i \in I} C_i)$ is open in X.
By De Morgan's Law, we have that $X \backslash (\bigcap_{i \in I} C_i) = \bigcup_{i \in I} (X \backslash C_i)$.
Since each C_i is closed we have that $X \backslash C_i$ is open in X.
Since X is a topological space, arbitrary unions of open sets is again open. Hence $X \backslash (\bigcap_{i \in I} C_i)$ is open and we are done.

Part 3: Let $i = 1 \ldots n$ and C_i a finite number of closed sets in X. It suffices to show that $X \backslash (\bigcup_{i=1}^{n} C_i)$ is open in X.
By De Morgan's Law, we have that $X \backslash (\bigcup_{i=1}^{n} C_i) = \bigcap_{i=1}^{n} (X \backslash C_i)$.
Since each C_i is closed, we have that $X \backslash C_i$ is open in X.
Since X is a topological space, finite intersections of open sets are again open in X. Hence $X \backslash (\bigcup_{i=1}^{n} C_i)$ is open and we are done.

□

Theorem 1.5.4. *Let Y be a subspace of a topological space X. Then a set* $A \subseteq Y$ *is closed in* $Y \iff A = Y \cap B$ *for some closed set B closed in X.*

Proof.

$\implies$: Let Y be a subspace of X and $A \subseteq Y$ a closed set in Y. By definition, $X \backslash A$ is open in Y. We can therefore write $Y \backslash A = Y \cap U$ for some set U open in X. This implies that $A = Y \cap (X \backslash U)$. Since U is open in X, $X \backslash U$ is closed in X as required.

$\Longleftarrow$: Let $A = Y \cap B$ for some closed set B in X. Since B is closed in X, we have that $X \backslash B$ is open in X. Hence $Y \backslash A = Y \cap (X \backslash B)$ is open in the subspace topology on Y.

□

Definition 1.5.5. *Let X be a topological space and $A \subseteq X$ a subset. We define the* **interior** *of A to be*

$$Int(A) = \bigcup_{U \subseteq A, U \in \tau} U$$

We define the **closure** *of A to be*

$$\bar{A} = \bigcap_{C \supseteq A,\ C \text{ closed}} C$$

We define the **boundary** *of A to be*

$$\partial A = \bar{A} \setminus Int(A)$$

Remark. *Int A is the biggest open set contained in A and A is open $\iff$ A = Int(A).*
$\bar{A}$ is the smallest closed set containing A and A is closed $\iff$ $A = \bar{A}$.

Proposition 1.5.6. *Let X be a topological space and $A \subseteq$ a subset. Then*

$$\bar{A} = X \setminus (Int(X \backslash A))$$

Proof. $\bar{A} = X \backslash (X \backslash \bar{A}) = X \backslash (X \backslash (\bigcap_{C \supseteq A, \text{ C closed}} C))$
$= X \backslash (\bigcup_{C \supseteq A, \text{ C closed}} (X \backslash C))$

Now let $U = X \backslash C$. By definition, U is open in X.

$= X \backslash (\bigcup_{U \subseteq X \backslash A, \text{ U open}} U)$
$= X \backslash (Int(X \backslash A))$ □

Example 1.5.7. *Consider $X \neq \varnothing$ endowed with the trivial topology and $A \subseteq X$ a subset. Then we have:*

$$Int(A) = \begin{cases} \varnothing & \text{if } A \subsetneq X \\ X & \text{if } A = X \end{cases}$$
$$\bar{A} = \begin{cases} \varnothing & \text{if } A = \varnothing \\ X & \text{if } A \neq \varnothing \end{cases}$$
$$\partial A = \begin{cases} \varnothing & \text{if } A = X, \varnothing \\ X & \text{if otherwise} \end{cases}$$

Example 1.5.8. *Let X be an infinite set endowed with the finite complement topology τ_f. Then we have:*

$$Int(A) = \begin{cases} A & \text{if } |X \backslash A| < \infty \\ \varnothing & \text{if } |X \backslash A| = \infty \end{cases}$$
$$\bar{A} = \begin{cases} A & \text{if } |A| < \infty \\ X & \text{if } |A| = \infty \end{cases}$$
$$\partial A = \begin{cases} X \backslash A & \text{if } |X \backslash A| < \infty \\ A & \text{if } |A| = \infty \\ X & \text{if } |A| < \infty, |X \backslash A| = \infty \end{cases}$$

Example 1.5.9. *Consider $\mathbb{R}$ endowed with the standard topology and $A = \mathbb{Q} \subseteq \mathbb{R}$ a subset. Then*

$$\forall x \in \mathbb{R}, \forall \varepsilon > 0, (x - \varepsilon, x + \varepsilon) \nsubseteq A \implies Int(A) = \varnothing$$

Hence we see that

$$\bar{A} = \mathbb{R} \backslash (Int(\mathbb{R} \backslash \varnothing)) = \mathbb{R}$$
$$\partial A = \mathbb{R} \backslash \varnothing = \mathbb{R}$$

Theorem 1.5.10. *Consider the topological space X with subspace $Y \subseteq X$ and $A \subseteq Y$ a subset. Denote $\bar{A}$ as the closure of A in X and $\tilde{A}$ as the closure of A in Y. Then*

$$\tilde{A} = \bar{A} \cap Y$$

Proof.

$\underline{\tilde{A} \subseteq \bar{A} \cap Y}$: Since $\bar{A}$ is closed in X, we have that $\bar{A} \cap Y$ is closed in Y. Now, since $A \subseteq \bar{A} \cap Y$ and, by definition, $\tilde{A}$ equals the intersection of all closed subsets of Y containing A, we have that $\tilde{A} \subseteq \bar{A} \cap Y$.

$\underline{\bar{A} \cap Y \subseteq \tilde{A}}$: $\tilde{A}$ is closed in Y hence $\tilde{A} = C \cap Y$ for some set C closed in X. Then C is a closed set of X containing A and, because $\bar{A}$ is the intersection of all such closed sets, we see that $barA \subseteq C$ hence $\bar{A} \cap Y \subseteq C \cap Y = \tilde{A}$.

□

Example 1.5.11. *Consider* $X = \mathbb{R}^2, Y = \mathbb{Q}^2, A = \mathbb{Z}^2$.
We want to find $\overline{A_{\mathbb{Q}}}$. *We first start by finding* $\bar{A}$ *(closure in* $\mathbb{R}^2$*).*
$\mathbb{R}^2 \backslash A$ *is open as we can always squeeze an open ball around any point.*
$\implies$ *A is closed* $\implies \bar{A} = A = \mathbb{Z}^2$.
Now $\overline{A_{\mathbb{Q}}} = \bar{A} \cap \mathbb{Q}^2 = \mathbb{Z}^2 \cap \mathbb{Q}^2 = \mathbb{Z}^2$.

Example 1.5.12. *Consider* $X = \mathbb{R}$, $(0,1] \subseteq \mathbb{R}$ *a subspace and* $A = (0, \frac{1}{2}) \subseteq (0,1] \subseteq \mathbb{R}$.
$\overline{A_{\mathbb{R}}} = [0, \frac{1}{2}] \implies \overline{A_{(0,1]}} = [0, \frac{1}{2}] \cap (0,1] = (0, \frac{1}{2}]$.

Example 1.5.13. *Consider* $S = \{\frac{1}{n} \mid n \geq 1\} \subseteq (0,1] \subseteq \mathbb{R}$.
First we calculate $\overline{S_{\mathbb{R}}} = \mathbb{R} \backslash Int(\mathbb{R} \backslash S)$.
$Int(\mathbb{R} \backslash S) = (\mathbb{R} \backslash S) \backslash \{0\} = \mathbb{R} \backslash (S \cup \{0\})$.
Hence, $\overline{S_{\mathbb{R}}} = S \cup \{0\}$.
Now, $\overline{S_{(0,1]}} = \overline{S_{\mathbb{R}}} \cap (0,1] = S \implies S$ *is closed in* $(0,1]$.

Theorem 1.5.14. *Let X be a topological space,* $A \subseteq X, x \in X$.

1. $x \in \bar{A} \iff$ *every neighbourhood of x intersects A*
2. *Let* $\mathcal{B}$ *be a topology basis for* τ. *Then* $x \in \bar{A} \iff \forall B$ *basic neighbourhood of x, B intersects A.*
3. $x \in \partial A \iff \forall$ *neighbourhood U of x, U intersects both A and* $X \backslash A$.

Proof.

Part 1:

$\implies$: Let $x \in \bar{A}$ and assume there exists a neighbourhood U_0 of x which does not intersect A.

Necessarily, $U_0 \subseteq X \backslash A \implies A \subseteq X \backslash U_0 = V$. Since U_0 is an open set, V by definition is closed. Now by the definition of the closure of A, we have that $x \in \bar{A} \subseteq V$. Hence $x \in V \cap U = \varnothing$ which is a contradiction.

$\impliedby$: Now consider $x \in X$ and assume $\forall U$ neighbourhood of x, $U \cap A \neq \varnothing$. Let C be any closed set such that $C \supseteq A$. Assume by contradiction that $x \notin C$. Let $x \in U = X \backslash C$ for some open set U. We have that $U = X \backslash C \subseteq X \backslash A$ which cannot intersect A. This contradicts that all neighbourhoods of x intersect A.

Part 2:

$\implies$: This is trivial by the fact that Part 1. applies to all neighbourhoods including basic neighbourhoods.

$\impliedby$: Let $\mathcal{B}$ be a topology basis for τ and assume that $\forall B$ basic neighbourhoods of a point x, $B \cap A \neq \varnothing$. It suffices to show that since all basic neighbourhoods of x intersect A, all neighbourhoods of x must intersect A; we can then apply Part 1. By Lemma 1.2.8, we know that all open sets consist of arbitrary unions of basic open sets. Hence any open neighbourhood U of x must intersect A since it is the union of basic neighbourhoods which intersect A.

Part 3: Assume that $x \in \partial A = \bar{A} \backslash Int(A), x \in \bar{A} \iff$ every neighbourhood of x intersects A. $x \notin Int(A) \iff$ every neighbourhood of x intersects $X \backslash A$.

□

Example 1.5.15. *Consider $X = \mathbb{R}_l$ (i.e $\mathbb{R}$ endowed with the lower limit topology) generated by $\{[a, b) \mid -\infty < a < b < \infty\}$ and $A = (0, 1)$. We want to compute $\bar{A}$.*
Firstly, we see that $x \in (0, 1) \implies x \in \bar{A}$. We can also see that if $x \notin [0, 1] \implies x \notin \bar{A}$.
We now need to check if $0, 1 \in \bar{A}$.
Let $[a, b)$ be a basic neighbourhood of zero. If $a = 0$, we have $[0, b) \cap (0, 1) \neq \varnothing$. If $a < 0, b > 0$ then again, $[a, b) \cap (0, 1) \neq \varnothing$. Hence $0 \in \bar{A}$.
Consider $[1, 2)$ a basic neighbourhood of 1. This contains no points of A and hence $1 \notin \bar{A} \implies \bar{A} = [0, 1)$.

Definition 1.5.16. *Let X be a topological space, $A \subseteq X$ a subset and $x \in X$. x is a **limit point** of A ($x \in A'$) if $\forall\, U$ neighbourhood of x, $U \cap (A \backslash \{x\}) \neq \varnothing$. (i.e every neighbourhood of x contains a point of A other than x.)*

Theorem 1.5.17. *Let X be a topological space and $A \subseteq X$ a subspace, then $\bar{A} = A \cup A'$ (the union is not necessarily disjoint).*

Proof.

$\bar{A} \subseteq A \cup A'$: Let $x \in \bar{A}$. We show that necessarily, $x \in A'$. Suppose $x \notin \bar{A} \backslash A$. Since $x \in \bar{A}$, every neighbourhood U of x intersects A at say $y \in U \cap A$. But $y \in A$ and $x \in A \implies y \neq x$. Hence $U \cap (A \backslash \{x\}) \neq \varnothing \implies x \in A'$.

□

$A \cup A' \subseteq \bar{A}$: Let $x \in A \cup A'$. Then $x \in A$ or $x \in A'$. First, assume $x \in A$. By definition, $x \in \bar{A}$. Now assume $x \in A'$. By definition, if x is a limit point of A, then every neighbourhood of x intersects A in a point other then x. Hence it must also be in $\bar{A}$.

Corollary 1.5.18. *A subset A of a topological space is closed $\iff$ $A' \subseteq A$.*

Proof. $A \subseteq X$ closed $\iff A = \bar{A} \iff A = A \cup A' \iff A' \subseteq A$. □

Definition 1.5.19. *Let X be a topological space, $\{x_n\} \subseteq X$ a sequence and $x \in X$. We say*

$$\{x_n\} \underset{n \to \infty}{\longrightarrow} x$$

if $\forall\, U$ neighbourhood of x, $\exists\, N \in \mathbb{N}$ such that $\forall\, n \geq N, x_n \in U$.

Example 1.5.20. *Let X be equipped with the trivial topology, $\{x_n\} \subseteq X$ a sequence, $x \in X$. There exists only one neighbourhood of x, namely $U = X$. Indeed, $x_n \in U \,\forall\, n \geq 1$. Hence $x_n \to x$.*

Proposition 1.5.21. *Let $X = \mathbb{R}$ endowed with τ_f the finite complement topology and let $x_n = (-1)^n$. We claim that $x_n \nrightarrow 1$.*

Proof. Take $U = \mathbb{R} \backslash \{-1\}$ a neighbourhood of 1. Now, $x_n = 1$ for n odd. No matter how big we choose N, we will always have that $x_n \notin U$ for odd n. Hence $x_n \nrightarrow 1$. □

Definition 1.5.22. *A topological space X is called* ***Hausdorff*** *or* $\mathbf{T_2}$ *if* $\forall\, x \neq y \in X, \exists\, U_1, U_2$ *neigbourhoods of* x *and* y *respectively such that* $U_1 \cap U_2 = \varnothing$.

Theorem 1.5.23. *Any finite set in a Hausdorff space is necessarily closed.*

Proof. Let X be a Hausdorff space and $A \subseteq X$ a finite subset. It is sufficient to show that any singleton in X is closed since if A is closed then it is the finite union of singletons.
Let $S = \{x\} \subseteq X$ be a singleton. Since X is Hausdorff, we have that $\forall\, y \in X, \exists\, U_y$ neighbourhood of y such that $x \notin U_y$.
Then $X \backslash S = \bigcap_{y \neq x} U_y$ is open $\implies \{x\}$ is closed. □

Example 1.5.24. $\mathbb{R}$ *equipped with the standard topology is a Hausdorff space. Indeed, given any* $x \neq y \in \mathbb{R}$*, we can always find* $\varepsilon, \delta > 0$ *such that* $(x - \varepsilon, x + \varepsilon) \cap (y - \delta, y + \delta) = \varnothing$.

Example 1.5.25. *A space X equipped with the discrete topology is a Hausdorff space. Indeed, given any* $x \neq y \in X$ *we can take* $U_x = \{x\}$ *and* $U_y = \{y\}$ *and hence* $U_x \cap U_y = \varnothing$.

Proposition 1.5.26. *Consider* $X = \mathbb{R}$ *equipped with the finite complement topology* τ_f*. Then X is not a Hausdorff space.*

Proof. $\tau_f = \{U \,|\, \mathbb{R} \backslash U \text{ is finite}\} \cup \{\varnothing\}$.
If U_1 and U_2 are neighbourhoods of two points $x \neq y \in \mathbb{R}$ then necessarily, $U_1 = \mathbb{R} \backslash \{x_1, x_2, \ldots, x_n\}, U_2 = \mathbb{R} \backslash \{y_1, y_2, \ldots, y_n\}$. Now, $U_1 \cap U_2 = \mathbb{R} \backslash \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\} \neq \varnothing$. Hence $\mathbb{R}$ equipped with the finite complement topology is not a Hausdorff space. □

Theorem 1.5.27. *Let X be a Hausdorff space and* $\{x_n\} \subseteq X$ *a sequence in X. Then* $\{x_n\}$ *converges to at most one element of X.*

Proof. Assume, to obtain a contradiction, that $x_n \to x, y$ with $x \neq y$. Let U_1, U_2 be two neighbourhoods of x, y such that $U_1 \cap U_2 = \varnothing$, the existence of which is guaranteed by the fact that X is a Hausdorff space.
By hypothesis, we have that:

$$\exists N_1 \text{ such that } \forall\, n \geq N_1, x_n \in U_1$$
$$\exists N_2 \text{ such that } \forall\, n \geq N_2, x_n \in U_2$$

Now take $N := max\{N_1, N_2\}$. We have that

$$\forall\, n \geq N, x_n \in U_1, x_n \in U_2 \implies x_n \in U_1 \cap U_2 = \varnothing$$

This is a contradiction and hence x_n must converge to a single limit point. □

1.6 Continuous functions

Definition 1.6.1. *Let X and Y be two topological spaces.* $f : X \to Y$ *is continuous if*

$$\forall V \subseteq Y \text{ open }, U = f^{-1}(V) \text{ is open}$$

Proposition 1.6.2. *If* $\mathcal{B}$ *is a topology basis on* Y *and* $\forall B \in \mathcal{B}, f^{-1}(B)$ *is open in* X *then f is continuous.*

Proof. Let V be an open set in Y. Then

$$f^{-1}(V) = f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$$

Since, by assumption, $f^{-1}B_i$ is open, we have $f^{-1}(B_i) = U_i$ for some set U_i open in X. Hence

$$\bigcup_{i \in I} f^{-1}(B_i) = \bigcup_{i \in I} U_i = U$$

Now since each U_i is open in X, we have that U is open in X and hence f is a continuous function. □

Example 1.6.3. *Consider the function*

$$f : \mathbb{R} \to \mathbb{R}_l$$
$$x \mapsto x$$

where $\mathbb{R}$ *denotes* $\mathbb{R}$ *equipped with the standard topology and* $\mathbb{R}_l$ *denotes* $\mathbb{R}$ *equipped with the lower limit topology.*
$V = [0, 1)$ *is indeed open in* $\mathbb{R}_l$ *yet* $f^{-1}([0, 1)) = [0, 1)$ *is not open in* $\mathbb{R}$*. Hence f is not continuous.*

Now consider the function

$$g : \mathbb{R}_l \to \mathbb{R}$$
$$x \mapsto x$$

$U = (a, b)$ *for some* $a, b \in \mathbb{R}$ *is indeed open in* $\mathbb{R}$*.* $f^{-1}((a, b)) = (a, b)$ *which is open in* $\mathbb{R}_l$*. Hence, g is continuous.*

Example 1.6.4. *Consider the function*

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto |x| \end{aligned}$$

where $\mathbb{R}$ *is equipped with the standard topology. This is a continuous function.*

Example 1.6.5. *Let* X *be a topological space and* $Y \subseteq X$ *a subspace. The function*

$$\begin{aligned} i : Y &\to X \\ y &\mapsto y \end{aligned}$$

is continuous. The subspace topology is the coarsest topology that makes i *continuous.*

Example 1.6.6. *Let* X *and* Y *be two topological spaces. The two functions*

$$\begin{aligned} \pi_1 : X \times Y &\to X \\ (x, y) &\mapsto x \end{aligned}$$

$$\begin{aligned} \pi_2 : X \times Y &\to Y \\ (x, y) &\mapsto y \end{aligned}$$

are continuous. The product topology is the coarsest topology that makes π_1, π_2 *continuous.*

Theorem 1.6.7. *Let* $f : X \to Y$ *be a map between two topological spaces* X *and* Y*. The following are equivalent*

1. f *is continuous*
2. $\forall\, A \subseteq X, f(\overline{A}) \subseteq \overline{f(A)}$
3. $\forall\, B \subseteq Y$ *closed,* $f^{-1}(B)$ *is closed*
4. $\forall\, x \in X, V$ *neighbourhood of* $f(x) \in Y, \exists\, U$ *neighbourhood of* x *such that* $f(U) \subseteq V$

Proof. We shall prove the theorem in the order $1 \implies 2 \implies 3 \implies 1$, $1 \implies 4 \implies 1$.

$1 \implies 2$: Assume that f is continuous and let $y \in f(\overline{A})$. Then $\exists\, x \in \overline{A}$ such that $f(x) = y$.
$\implies \forall U$ neighbourhood of $x, U \cap A \neq \varnothing$.
Let V be any neighbourhood of y such that $U = f^{-1}(V)$ is a neighbourhood of x, the existence of which is guaranteed by the assumption that f is continuous.
Now $U \cap A \neq \varnothing \implies f(U) \cap f(A) \neq \varnothing \subseteq V \cap f(A)$.
$\implies y \in f(A) \implies y \in \overline{f(A)}$.

$2 \implies 3$: Let B be a closed set in Y and let $A = f^{-1}(B)$. We want to show that A is closed in X hence it suffices to show that $\overline{A} = A$.
We have that $f(A) = f(f^{-1}(B)) \subseteq B$. Now let $x \in \overline{A}$.
$\implies f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B$
$\implies x \in f^{-1}(B) = A$
$\implies \overline{A} \subseteq A \implies \overline{A} = A$.

$3 \implies 1$: Let $U \subseteq Y$ be open. We have that $f^{-1}(U) = X \backslash f^{-1}(Y \backslash U)$.
By assumption, we have that $f^{-1}(Y \backslash U) closed \implies f^{-1}(U)$ is open.

$1 \implies 4$: Consider $x \in X, V$ neighbourhood of $f(x) \in Y$. Then $U = f^{-1}(V)$ is a neighbourhood of x.

$4 \implies 1$: Consider $V \subseteq Y$ open and take $U = f^{-1}(V), x \in U$. By assumption, $\exists\, U_x$ such that $f(U_x) \subseteq V$ is a neighbourhood of x.
Then $U = \bigcup_{x \in U} U_x$ is the union of open sets, hence U is open.

□

Definition 1.6.8. *Consider a function $f : X \to Y$. We say that f is a* ***homeomorphism*** *if f is a continuous bijection whose inverse is also continuous.*
We say that two topological spaces X and Y are ***homeomorphic*** *if there exists a homeomorphism $f : X \to Y$.*

Example 1.6.9. *The function*

$$\begin{aligned} id : X &\to X \\ x &\mapsto x \end{aligned}$$

is a homeomorphism.

Example 1.6.10. *Consider* $Y \subseteq X$ *a subspace. The function*

$$\begin{aligned} i : Y &\to X \\ y &\mapsto y \end{aligned}$$

is not a homeomorphism as i *is not bijective.*

Example 1.6.11. *Consider the function*

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f(x) = 3x - 1 \end{aligned}$$

f *is bijective, continuous and its inverse* $f^{-1}(y) = \frac{y+1}{3}$ *is also continuous, hence it is a homeomorphism.*

Example 1.6.12. *Consider the function*

$$\begin{aligned} f : [0, 1) &\to S^1 \\ t &\mapsto f(t) = (cos(2\pi t), sin(2\pi t)) \end{aligned}$$

where $[0, 1)$ *is endowed with the subspace topology from* $\mathbb{R}$ *and* S^1 *is the unit circle endowed with the subspace topology from* $\mathbb{R}^2$*.*
We have that f *is bijective and continuous. However,* $f([0, \frac{1}{4})) \subseteq S^1$ *is not an open set of* S^1*. To see this, consider* $f(0) \in S^1$*. We can find no open set* $U \subseteq \mathbb{R}^2$ *such that* $f(0) \in (U \cap S^1)$*. Hence* f *is not a homeomorphism.*

Theorem 1.6.13. *Consider a function* $f : X \to Y$ *with* $X = \bigcup_{i\in I} U_i, \forall\, i \in I, U_i$ *is open. If* $\forall\, i \in I, f|_{U_i} : U_i \to Y$ *is continuous then* f *is continuous.*

Proof. Let $V \subseteq Y$ be open. $f^{-1}(V) = f^{-1}(V) \cap (\bigcup_{i\in I} U_i) = \bigcup_{i\in I}(f^{-1}(V) \cap U_i) = \bigcup_{i\in I}(f|_{U_i})^{-1}(V)$. By assumption, we have that $\forall\, i \in I, (f|_{U_i})^{-1}(V)$ is open in $U_i \implies$ V is open in X $\implies$ f is continuous. □

Theorem 1.6.14. *(Pasting Lemma)*
Let $X = A \cup B$ where A and B are closed in X. Consider $f : A \to Y, g : B \to Y$ such that $\forall\, x \in A \cap B, f(x) = g(x)$. Define

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

where $h : X \to Y$, then h is a continuous function.

Proof. It suffices to show that $\forall\, C \subseteq Y closed, h^{-1}(C) \subseteq X$ is closed.
$h^{-1}(C) = (h^{-1}(C) \cap A) \cup (h^{-1}(C) \cap B) = f^{-1}(C) \cup g^{-1}(C)$
Now, since f and g are continuous functions, we have that $f^{-1}(C)$ and $g^{-1}(C)$ are open in A and B respectively. Hence they are both closed in X and their union is also closed in X. This implies that h is a continuous function. □

Example 1.6.15. *Consider the functions*

$$\begin{aligned} f : [1, \infty) &\to \mathbb{R} \\ x &\mapsto x \end{aligned}$$

$$\begin{aligned} g : (-\infty, 1] &\to \mathbb{R} \\ x &\mapsto -(x - 2) \end{aligned}$$

We have that $[1, \infty) \cap (-\infty, 1] = \{1\}$ and $f(1) = g(1) = 1$.
Hence, $h(x) = |x - 1| + 1$ is continuous by the Pasting Lemma.

Example 1.6.16. *Consider the two functions*

$$\begin{aligned} f : [\sqrt{2}, \infty) \cap \mathbb{Q} &\to \mathbb{R} \\ x &\mapsto 1 \end{aligned}$$

$$\begin{aligned} g : (-\infty, \sqrt{2}] \cap \mathbb{Q} &\to \mathbb{R} \\ x &\mapsto 0 \end{aligned}$$

By definition of the subspace topology on $\mathbb{Q}$, $A = [\sqrt{2}, \infty) \cap \mathbb{Q}$ and $(-\infty, \sqrt{2}] \cap \mathbb{Q}$ are both closed in $\mathbb{Q}$. We also have that $A \cap B = \varnothing \implies f|_{A \cap B} = g|_{A \cap B}$. Hence h is continuous by the Pasting Lemma.

Proposition 1.6.17. *Let X, Y and Z be three topological spaces and consider the following functions*

$$f : X \to Y$$
$$g : Y \to Z$$

If f and g are both continuous then $g \circ f$ is also continuous.

Proof. Consider $U_z \subseteq Z$ a basic open set. We show that $(g \circ f)^{-1}(U_z)$ is open.
Since g is a continuous function, we have that $U_y = g^{-1}(U_z)$ is open for some $U_y \subseteq Y$.
Now, since f is also continuous, we have that $U_x = f^{-1}(U_y)$ is open for some $U_x \subseteq X$.
But $U_x = f^{-1}(U_y) = f^{-1}(g^{-1}(U_z)) = (g \circ f)^{-1}(U_z)$, where U_z is basic open in Z and U_x is open in X.
Hence $g \circ f$ is continous. □

1.7 The Product Topology

Example 1.7.1. *Consider*

$$\mathbb{R}^\omega = \prod_{i \in \mathbb{Z}} \mathbb{R} = \{(a_i)_{i \in \mathbb{Z}} \,|\, a_i \in \mathbb{R}\}$$

In this section, we will look at endowing $\mathbb{R}^\omega$ with a topology.

Definition 1.7.2. *The **box topology** on $X = \prod_{i \in I} X_i$ is the topology generated by the topology basis*

$$\mathcal{B} = \left\{ \prod_{i \in I} U_i \,\middle|\, \forall\, i, U_i \in \tau_{X_i} \right\}$$

Definition 1.7.3. *The **product topology** on $X = \prod_{i \in I} X_i$ is the topology generated by the topology basis*

$$\mathcal{B} = \left\{ \prod_{i \in I} U_i \,\middle|\, \forall\, U_i \in \tau_{X_i} \text{ and } \forall\, i \neq 1, \ldots, n\,, U_i = X_i \right\}$$

Remark.

- *The product topology is coarser than the box topology.*
- *If $|I| < \infty$ then the product topology is exactly the box topology.*

Example 1.7.4. *Consider $\mathbb{R}^\omega = \prod_{i \in \mathbb{Z}} \mathbb{R}$ and $A = \prod_{i \in \mathbb{Z}} (-1, 1) \subseteq \mathbb{R}^\omega$. Then A is open in the box topology however it is not open in the product topology. This is because for infintely many $i \in \mathbb{Z}$ the only open neighbourhood of $x \in \mathbb{R}$ is $\mathbb{R}$. It is obviously not possible to squeeze $\mathbb{R}$ inside of $(-1, 1)$ hence for infinitely many $i \in \mathbb{Z}$, there are points that are not open thus A is not open.*

Proposition 1.7.5. *Consider $X = \prod_{i \in \mathbb{N}} \{0, 1\}$ where $\{0, 1\}$ is equipped with the discrete topology. Then the box topology on X is discrete.*

Proof. Consider $x \in X$. Then $x = (x_1, x_2, x_3, \dots)$ for some $x_i \in \{0, 1\}$. Each x_i is open in the discrete topology on $\{0, 1\}$ and hence $\prod_{i \in \mathbb{N}} x_i$ is open in the box topology. □

Example 1.7.6. *Consider $\mathbb{R}^\omega$. We have that*

$$\mathcal{B} = \left\{ \prod_{i \leq n} (a_i, b_i) \times \prod_{i > n} \mathbb{R} \;\middle|\; \forall\, i, -\infty < a_i < b_i < \infty \right\}$$

is a basis for the product topology on $\mathbb{R}^\omega$ and

$$\mathcal{B} = \left\{ \prod_{i \geq 1} (a_i, b_i) \;\middle|\; \forall\, i, -\infty < a_i < b_i < \infty \right\}$$

is a basis for the box topology on $\mathbb{R}^\omega$.

Theorem 1.7.7. *Consider a topological space $X = \prod_{i \in I} X_i$, let $A_i \subseteq X_i$ be a subspace and $A = \prod_{i \in I} A_i \subseteq X$. Let τ be the product topology on A and τ' be the subspace topology induced on A from X endowed with the product topology. Then $\tau = \tau'$. The same holds for the box topology.*

Proof. We prove the theorem for the product topology. Consider the set

$$U = \prod_{i \leq n} U_i \times \prod_{i > n} X$$

where U_i a general basis element of X_i. We can see that

$$U \cap A = \left(\prod_{i \leq n} U_i \times \prod_{i > n} X \right) \cap \prod_{i \in I} A_i$$

is the general basis element for the subspace topology on A. Now

$$\begin{aligned} U \cap A &= \left(\prod_{i \leq n} U_i \times \prod_{i > n} X \right) \cap \prod_{i \in I} A_i \\ &= \prod_{i \leq n} (U_i \cap A_i) \times \prod_{i \geq n} (X \cap A_i) \\ &= \prod_{i \leq n} (U_i \cap A_i) \times \prod_{i \geq n} X \end{aligned}$$

where the last equation is the general basis element for the product topology on A. Since the two bases agree, the two topologies must be equal. □

Theorem 1.7.8. *Consider a topological space $X = \prod_{i \in I} X_i$ endowed with the product topology. If each X_i are Hausdorff spaces then X is a Hausdorff space. The same also applies to the product topology.*

Proof. We prove the theorem for the product topology. Let each X_i be Hausdorff spaces. Consider $x \neq y \in X$ and suppose, without loss of generality, they differ at only one index; say i $= i_0$. We want to show that there exists $U_x, U_y \subseteq X$ open neighbourhoods of x and y respectively such that $U_x \cap U_y = \varnothing$. Let

$$\begin{aligned} U_x &= U_{x_{i_0}} \times \prod_{i \neq i_0} X_i \\ U_y &= U_{y_{i_0}} \times \prod_{i \neq i_0} X_i \end{aligned}$$

where $U_{x_{i_0}}$, $U_{y_{i_0}}$ are open neighbourhoods of x_i and y_i respectively such that $U_{x_{i_0}} \cap U_{y_{i_0}} = \varnothing$, the existence of which is guaranteed by the fact that X_{i_0} is a Hausdorff space.
To show that $U_x \cap U_y = \varnothing$, suppose $\exists\, z \in U_x \cap U_y$. Then $z_{i_0} \in U_{x_{i_0}} \cap U_{y_{i_0}} = \varnothing$. This is obviously not possible, hence $U_x \cap U_y = \varnothing$. □

Theorem 1.7.9. *Consider a topological space* $X = \prod_{i\in I} X_i$ *endowed with the product topology and* $A_i \subseteq X_i$ *subsets. Then*

$$\overline{\prod_{i\in I} A_i} = \prod_{i\in I} \overline{A_i}$$

The same applies to the box topology.

Proof.

$\subseteq$: Let $x = (x_i)_{i\in I} \in \overline{A} = \overline{\prod_{i\in I} A_i}$. We want to show that $\forall\, i \in I, x_i \in \overline{A_i}$.
We first fix an index $i = i_0$ and let V_{i_0} be a neighbourhood of x_{i_0}.
Consider

$$V = V_{i_0} \times \prod_{i\neq i_0} X_i$$

a basic neighbourhood of $x \in X$.
By Part 1. of Theorem 1.5.14, we have

$$x \in \overline{A} \implies V \cap A \neq \varnothing \implies V_{i_0} \cap A_{i_0} \neq \varnothing$$

Since V_{i_0} is an arbitrary neighbourhood of x_{i_0}, we can again invoke Theorem 1.5.14 to see that $x_{i_0} \in \overline{A_{i_0}}$.

$\supseteq$: Let $x = (x_i)_{i\in I} \in \prod_{i\in I} \overline{A_i}$ and V a basic neighbourhood of x. By assumption, $x_i \in \overline{A_i}\ \forall\, i \in I$.
Then

$$V = \prod_{i\leq n} V_i \times \prod_{i>n} X_i$$

for some finite n. Let us first consider the case where $i \leq n$.
Since $x_i \in \overline{A_i}$, we can invoke Theorem 1.5.14 and

$$V_i \cap A_i \neq \varnothing \implies \left(\prod_{i\leq n} V_i\right) \cap \left(\prod_{i\leq n} A_i\right) = \prod_{i\leq n}(V_i \cap A_i) \neq \varnothing$$

We now consider the case where $i > n$.
Again, since $x_i \in \overline{A_i}$, we can invoke Theorem 1.5.14 and

$$X_i \cap A_i \neq \varnothing \implies \left(\prod_{i>n} X_i\right) \cap \left(\prod_{i>n} A_i\right) = \prod_{i>n}(X_i \cap A_i) \neq \varnothing$$

Now combining the two together, we get

$$V \cap A = \prod_{i \leq n} (V_i \cap A_i) \times \prod_{i > n} (X_i \cap A_i) \neq \varnothing$$

Since V is an arbitrary neighbourhood of x, we can once again invoke Theorem 1.5.14 to see that $x \in \overline{A}$.

□

Theorem 1.7.10. *Let $X = \prod_{i \in I} X_i$ be a topological space endowed with the product topology and A an arbitrary topological space. Consider the function*

$$\begin{aligned} f : A &\to X \\ a &\mapsto (f_i(a))_{i \in I} \end{aligned}$$

where $f_i : A \to X_i$ for each i. Then the function f is continuous if and only if each f_i is continuous.

Proof.

$\Longrightarrow$: Suppose that f is continuous. We want to show that each f_i is continuous. it suffices to show that if $U_i \subseteq X_i$ is a basic open neighbourhood then $f_i^{-1}(U_i)$ is open.
Consider π_i the projection of the product onto its i^{th} factor. By the definition of the product topology, π_i is continuous. Indeed, if

$$\begin{aligned} \pi_i : X &\to X_i \\ (x_j)_{j \in J} &\mapsto x_i \end{aligned}$$

and $U_i \subseteq X_i$ is basic open, we have that

$$\pi_i^{-1}(U_i) = U_i \times \prod_{j \neq i} X_j$$

which is open in X.
Since the composition of continuous functions is continuous, we have that $f_i = \pi_i \circ f$ is continuous.

$\Longleftarrow$: Now suppose that $\forall\, i \in I$, f_i is continuous. We have to show that given a basic open neighbourhood $U \subseteq X$, $f^{-1}(U)$ is open in A.

Consider the following basic open neighbourhood of the product topology on X

$$U = \prod_{i \leq n} U_i \times \prod_{i > n} X_i$$

where $n \in \mathbb{Z}_{\geq 0}$ and $U_i \subseteq X$ is basic open. It is easy to see that $a \in f^{-1}(U) \iff f_i(a) \in U_i \, \forall \, i$. Hence

$$\begin{aligned} f^{-1}(U) &= f_1^{-1}(U_1) \cap f_2^{-1}(U_2) \cap \cdots \cap f_n^{-1}(U_n) \cap \left(\bigcap_{i>n} f_i^{-1}(X_i) \right) \\ &= f_1^{-1}(U_1) \cap f_2^{-1}(U_2) \cap \cdots \cap f_n^{-1}(U_n) \cap \left(\bigcap_{i>n} A \right) \end{aligned}$$

but each $f_i^{-1}(U_i)$ is a subset of A, hence

$$f^{-1}(U) = f_1^{-1}(U_1) \cap f_2^{-1}(U_2) \cap \cdots \cap f_n^{-1}(U_n)$$

Now, $f^{-1}(U)$ is the finite intersection of open sets, hence it is open itself. Therefore f is continuous.

□

1.8 The Metric Topology

Definition 1.8.1. *Let X be a topological space. A* ***metric*** *on X is a function*

$$d : X \times X \to \mathbb{R}$$

satisfying the following three axioms

1. $\forall \, x, y \in X, d(x, y) \geq 0$ *and* $d(x, y) = 0 \iff x = y$
2. $\forall \, x, y \in X, d(x, y) = d(y, x)$
3. $\forall \, x, y, z \in X, d(x, y) + d(y, z) \geq d(x, z)$

Definition 1.8.2. *Let* $\varepsilon > 0$. *The* ***ε-ball centered at x*** *is the set*

$$B_d(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\}$$

In other words, it is the set of all points that are less than ε d-distance away from x.

Example 1.8.3. *Consider $X = \mathbb{R}^n$. We have the standard Euclidean metric on X:*

$$d_2(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

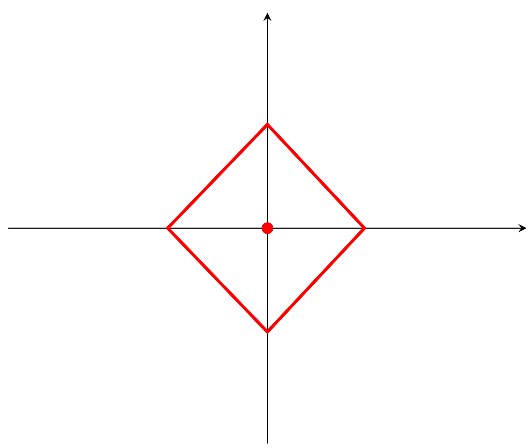
For $0 < p \leq \infty$, the following is also a metric on X

$$d_p(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^p \right)^{\frac{1}{p}}$$

Example 1.8.4. *Let $x, y \in \mathbb{R}^2$ and define*

$$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$$

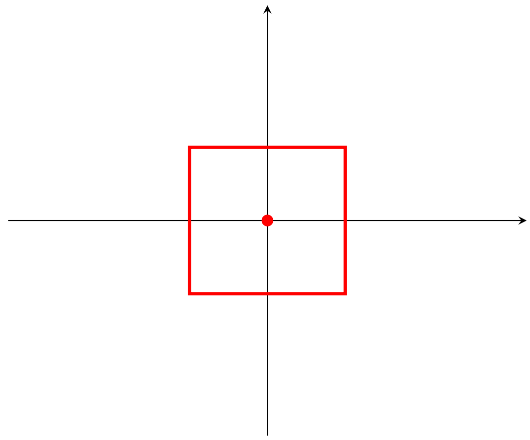
Then this is a metric on $\mathbb{R}^2$ and the open balls are diamond shaped:



Example 1.8.5. *Let $x, y \in \mathbb{R}^2$ and define*

$$d_\infty = max_{1 \leq i \leq n} |x_i - y_i|$$

Then this is a metric on $\mathbb{R}^2$ *and the open balls are square shaped:*



Proposition 1.8.6. *Let* X *be a set and* d *a metric on* X*. Then*

$$\mathcal{B} = \{B_d(x,r) \,|\, x \in X, r > 0\}$$

is a topology basis for the ***metric topology*** *on* X *induced by* d*.*

Proof. We prove that $\mathcal{B}$ satisfies the axioms of a basis.

1. $\forall\, x \in X, \exists\, B \in \mathcal{B}$ such that $x \in B$
   Let $x \in X$. Then $\forall\, r > 0, x \in B(x,r) \in \mathcal{B}$

2. $\forall\, B_1, B_2 \in \mathcal{B}, \forall\, x \in B_1 \cap B_2, \exists\, B_3 \subseteq B_1 \cap B_2$ such that $x \in B_3$

   Consider $x, y \in X$ and the open balls

   $$B_1 = B(x, r_1)$$
   $$B_2 = B(y, r_2)$$

   with $r_1, r_2 > 0$.
   Now consider $z \in B_1 \cap B_2$ and let $\varepsilon = min\{r_1 - d(z,x), r_2 - d(z,y)\}$.
   We claim that $B_3 = B(z,\varepsilon) \subseteq B_1 \cap B_2$.
   Let $t \in B(z,\varepsilon)$. Then $d(t,z) < \varepsilon$.
   By the properties of metrics, we have that

   $$\begin{aligned} d(x,t) &\leq d(x,z) + d(z,t) \\ &< d(x,z) + \varepsilon \\ &< d(x,z) + (r_1 - d(x,z)) \\ &< r_1 \end{aligned}$$

This implies that $t \in B_1$. By a similar argument on B_2, we have that

$$\begin{aligned} d(y,t) &\leq d(y,z) + d(z,t) \\ &< d(y,z) + \varepsilon \\ &< d(y,z) + (r_2 - d(y,z)) \\ &< r_2 \end{aligned}$$

Hence $t \in B_2 \implies t \in B_1 \cap B_2$.

□

Definition 1.8.7. *A topological space is said to be* ***metrisable*** *if there exists a metric d on the set X that induces the topology of X. A* ***metric space*** *is a metrisable space X together with a specific metric d that gives the topology of X.*

Theorem 1.8.8. *The topologies on $\mathbb{R}^n$ induced by the euclidean metric d_2 and the square metric d_∞ are the same as the product topology on $\mathbb{R}^n$.*

Proof. Let $x, y \in \mathbb{R}^2$. It is easy to see that

$$d_\infty(x,y) \leq d_2(x,y) \leq \sqrt{n} d_\infty(x,y)$$

Now, if $d_2(x,y) < \varepsilon$ then $d_\infty(x,y) < \varepsilon$. Hence $B_{d_2}(x,\varepsilon) \subseteq B_{d_\infty}(x,\varepsilon)$. Conversely

$$d_\infty(x,y) < \frac{\varepsilon}{\sqrt{n}} \implies d_2(x,y) < \sqrt{n}\, d_\infty(x,y) < \varepsilon$$

Hence $B_{d_\infty}(x,\varepsilon) \subseteq B_{d_2}(x,\varepsilon)$. Since we can excise a basic open d_2-ball inside any d_∞-ball and vice-versa, by Lemma 1.2.12, the topologies induced by the two metrics are equivalent.

It now suffices to show that the topology induced by the square metric is the same as the product topology on $\mathbb{R}$. Consider

$$B = \prod_{i=1}^{n} (a_i, b_i)$$

with $a, b \in \mathbb{R}$ a basic open neighbourhood for the product topology on $\mathbb{R}^n$. Consider $x = (x_1, \ldots, x_n) \in B$. $\forall\, i, \exists\, \varepsilon_i$ such that

$$(x_i - \varepsilon_i, x_i - \varepsilon_i) \subseteq (a_i, b_i)$$

Now take $\varepsilon = min\{\varepsilon_1, \dots, \varepsilon_n\}$ then it is easy to see that $B_{d_\infty}(x, \varepsilon) \subseteq B$. Hence the d_∞-topology is finer than the product topology.
Conversely, consider

$$B_{d_\infty}(x, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon)$$

where $\varepsilon > 0$ a basis element for the d_∞-topology on $\mathbb{R}^n$. We need to find a basic open neighbourhood B of the product topology such that $\forall\, y \in B_{d_\infty}(x, \varepsilon), y \in B \subseteq B_{d_\infty}(x, \varepsilon)$. However this is trivial as $B_{d_\infty}(x, \varepsilon)$ is itself a basic open neighbourhood of the product topology. Hence the product topology is finer than the d_∞-topology meaning the two topologies are equal. □

Theorem 1.8.9. *Let $\bar{d}(a, b) = min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. Let $x, y \in \mathbb{R}^\omega$ and consider*

$$D(x, y) = sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$

Proof. We first prove that D is a metric.

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$
   We have that

$$\frac{\bar{d}(x_i, y_i)}{i} \geq 0$$
$$\implies D(x, y) = sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\} \geq 0$$

   A similar argument also confirms the second part.

2. $d(x, y) = d(y, x)$
   We have that

$$\frac{\bar{d}(x_i, y_i)}{i} = \frac{\bar{d}(y_i, x_i)}{i}$$
$$\implies D(x, y) = sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\} = sup\left\{\frac{\bar{d}(y_i, x_i)}{i}\right\} = D(y, x)$$

3. $d(x, z) \leq d(x, y) + d(y, z)$
   We have that

$$\frac{\bar{d}(x_i, z_i)}{i} \leq \frac{\bar{d}(x_i, y_i)}{i} + \frac{\bar{d}(y_i, z_i)}{i} \leq D(x, y) + D(y, z)$$
$$\implies sup\left\{\frac{\bar{d}(x_i, z_i)}{i}\right\} \leq D(x, y) + D(y, z)$$

We now show that the D-topology on $\mathbb{R}^\omega$ is equivalent to the product topology. It suffices to show that we can squeeze an open neighbourhood of the product topology around every point inside an open D-ball and vice versa. We can then invoke Lemma 1.2.12 to show that the topologies are equivalent. Consider the ball

$$B_D(x, \varepsilon), x \in \mathbb{R}^\omega, \varepsilon > 0$$

We must find an open neighbourhood of the product topology $U \subseteq \mathbb{R}^\omega$ such that $U \subseteq B_D(x, \varepsilon)$. Take

$$U = \prod_{i=1}^{k} B_{\bar{d}_i}\left(x_i, \frac{\varepsilon}{2}\right) \times \prod_{i \geq k} \mathbb{R} \tag{1.1}$$

an open neighbourhood of $x \in \mathbb{R}^\omega$.
Let $y \in U$. We must show that $D(x, y) < \varepsilon$. Hence it suffices to find k such that

$$sup_{i \leq i \leq k}\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\} + sup_{i > k}\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\} < \varepsilon$$

Now,

$$sup_{i \leq i \leq k}\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\} + sup_{i > k}\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\} \leq sup_{i \leq i \leq k}\left\{\frac{r}{2i}\right\} + sup_{i > k}\left\{\frac{1}{i}\right\}$$

Hence we require that

$$\frac{r}{2} + \frac{1}{k} < r$$

We can then see that

$$\frac{1}{k} < r - \frac{r}{2}$$
$$\implies \frac{1}{k} < \frac{r}{2}$$
$$\implies k > \frac{2}{r}$$

We can therefore take $k = \lceil \frac{2}{r} \rceil$ and plug it into Equation 1.1 to get the desired neighbourhood. This shows that the product topology is finer than the D-topology.
Conversely, consider the basic open neighbourhood U of $x \in \mathbb{R}^\omega$

$$U = \prod_{i \leq n} U_i \times \prod_{i > n} \mathbb{R}$$

Where each U_i is an open neighbourhood of $x_i \in \mathbb{R}$.
Let $B_{\bar{d}_i}(x_i, r)$ be the open ball defined by $\bar{d}$ centered at x_i with radius r.
Since $\bar{d}$ is a metric on $\mathbb{R}$, we have that

$$\exists\, \varepsilon_i > 0 \text{ such that } B_{\bar{d}_i}(x_i, \varepsilon_i) \subseteq U_i$$

Let

$$\varepsilon = min_{1 \leq i \leq n} \left\{ \frac{\varepsilon_i}{i} \right\}$$

We claim that $B_D(x, \varepsilon) \subseteq U$ where $B_D(x, \varepsilon)$ is the D-ball centered at x of radius r.
Consider $y \in B_D(x, \varepsilon)$. Then $D(x, y) < \varepsilon$.

$$\implies sup_{i \leq i \leq n} \left\{ \frac{\bar{d}_i(x_i, y_i)}{i} \right\} < \varepsilon$$
$$\implies \frac{\bar{d}_i(x_i, y_i)}{i} < \varepsilon,\ \forall\, 1 \leq i \leq n$$
$$\implies \frac{\bar{d}_i(x_i, y_i)}{i} < \frac{\varepsilon_i}{i},\ \forall\, 1 \leq i \leq n$$
$$\implies \bar{d}_i(x_i, y_i) < \varepsilon_i,\ \forall\, 1 \leq i \leq n$$
$$\implies y_i \in B_{\bar{d}_i}(x_i, \varepsilon_i) \subseteq U_i,\ \forall\, 1 \leq i \leq n$$
$$\implies y \in U$$

This shows that the D-topology is finer than the product topology. Hence by Lemma 1.2.12, the two topologies are equivalent. □

Theorem 1.8.10. *Consider topological spaces X_i each equipped with a metric d_i. Let $\bar{d}_i(x, y) = min\{1, d_i(x_i, y_i)\}$ where $x_i, y_i \in X_i$. Then*

$$D(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \bar{d}_i(x_i, y_i)$$

is a metric for the product topology on $X = \prod_{i \in I} X_i$

Proof. The proof that D is a metric is left as an exercise to the reader.
We show that the D-topology on X is equivalent to the product topology on X. Hence it suffices to show that every D-ball contains a basic open neighbourhood of the product topology and vice versa. We can then invoke Lemma 1.2.12 to arrive at the desired result.
Consider the D-ball $B_D(x, \varepsilon)$. We wish to find a basic open neighbourhood U of the product topology such that $U \subseteq B_D(x, \varepsilon)$.
We shall take

$$U = \prod_{1 \leq i \leq n} B_{\bar{d}_i} \left(x_i, \frac{\varepsilon}{2} \right) \times \prod_{i > n} X_i \tag{1.2}$$

where x_i is the i^{th} coordinate of x.
Now consider $y \in U$, we need to show that $D(x, y) < \varepsilon$.

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \bar{d}_i(x_i, y_i) = \sum_{i=1}^{n} \frac{1}{2^i} \bar{d}_i(x_i, y_i) + \sum_{i=n+1}^{\infty} \frac{1}{2^i} \bar{d}_i(x_i, y_i) < \varepsilon$$

We therefore need to find n such that the above inequality holds.

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{1}{2^i}\bar{d}_i(x_i, y_i) + \sum_{i=n+1}^{\infty} \frac{1}{2^i}\bar{d}_i(x_i, y_i) \\
&< \frac{\varepsilon}{2}\sum_{i=1}^{n} \frac{1}{2^i} + \sum_{i=n+1}^{\infty} \frac{1}{2^i} \\
&= \frac{\varepsilon}{2}\left(\sum_{i=0}^{n} \frac{1}{2^i} - 1\right) + \sum_{i=0}^{\infty} \frac{1}{2^i} - \sum_{i=0}^{n} \frac{1}{2^i} \\
&= \frac{\varepsilon}{2}\left(\frac{1 - \frac{1}{2}^{n+1}}{1 - \frac{1}{2}} - 1\right) + \frac{1}{1 - \frac{1}{2}} - \frac{1 - \frac{1}{2}^{n+1}}{1 - \frac{1}{2}} \\
&= \frac{\varepsilon}{2}\left(2 - 2\frac{1}{2}^{n+1} - 1\right) + 2 - 2 + 2\frac{1}{2}^{n+1} \\
&= \frac{\varepsilon}{2}\left(1 - 2\frac{1}{2}^{n+1}\right) + 2\frac{1}{2}^{n+1} \\
&= \frac{\varepsilon}{2} - \varepsilon\frac{1}{2}^{n+1} + 2\frac{1}{2}^{n+1} \\
&= \frac{\varepsilon}{2} + \frac{1}{2}^{n+1}(2 - \varepsilon)
\end{aligned}$$

Now assume $0 < \varepsilon < 1$. We are allowed to do this since if $\varepsilon > 1$, the neighbourhood we are constructing will still be valid inside the bigger D-ball. We now require

$$\begin{aligned}
&\frac{\varepsilon}{2} + \frac{1}{2}^{n+1}(2-\varepsilon) < \varepsilon \\
&\implies \frac{1}{2}^{n+1} < \frac{\varepsilon}{2} \\
&\implies \frac{1}{2}^{n+1} < \frac{1}{2} \\
&\implies (n+1)\log_{\frac{1}{2}}\left(\frac{1}{2}\right) > \log_{\frac{1}{2}}\left(\frac{1}{2}\right) \\
&\implies (n+1) > \log_{\frac{1}{2}}\left(\frac{1}{2}\right) \\
&\implies n > \log_{\frac{1}{2}}\left(\frac{1}{2}\right) - 1 \\
&\implies n > 0
\end{aligned}$$

Hence we can substitute $n = 1$ into Equation 1.2 to give us the desired result.
Now consider the basic open neighbourhood U. We need to find a D-ball $B_D(x, \varepsilon)$ such that $B_D(x, \varepsilon) \subseteq U$. We consider

$$U = \prod_{i=1}^{n} U_i \times \prod_{i>n} X_i$$

where U_i is a basic open neighbourhood $U_i \subseteq X_i$.
Since each $\bar{d}_i$ is a metric on X_i, we have that

$$\exists\, \varepsilon_i \text{ such that } B_{\bar{d}_i}(x_i, \varepsilon_i) \subseteq U_i$$

Now take

$$\varepsilon = min_{1 \leq i \leq n}\left\{\frac{\varepsilon_i}{2^i}\right\}$$

Now consider $y \in B_D(x, \varepsilon)$, we have to show that $y \in U$.

$$
\begin{aligned}
&D(x,y) < \varepsilon \\
&\implies \sum_{i=1}^{\infty} \frac{1}{2^i}\bar{d}_i(x_i,y_i) < \varepsilon \\
&\implies \sum_{i=1}^{n} \frac{1}{2^i}\bar{d}_i(x_i,y_i) + \sum_{i=n}^{\infty} \frac{1}{2^i}\bar{d}_i(x_i,y_i) < \varepsilon \\
&\implies \sum_{i=1}^{n} \frac{1}{2^i}\bar{d}_i(x_i,y_i) < \varepsilon \\
&\implies \frac{1}{2^i}\bar{d}_i(x_i,y_i) < \varepsilon,\ \forall 1 \le i \le n \\
&\implies \frac{1}{2^i}\bar{d}_i(x_i,y_i) < \frac{\varepsilon_i}{2^i},\ \forall 1 \le i \le n \\
&\implies \bar{d}_i(x_i,y_i) < \varepsilon_i,\ \forall 1 \le i \le n
\end{aligned}
$$

Hence $y_i \in B_{\bar{d}_i}(x_i,\varepsilon_i)\ \forall 1 \le i \le n \implies y \in U$. This shows that the D-topology is finer than the product topology, hence by Lemma 1.2.12, the two topologies are equal. □

1.9 The Quotient Topology

Definition 1.9.1. *Consider two topological spaces X and Y and $p : X \to Y$ a surjective map. The map p is said to be a* ***quotient map*** *provided a subset $U \subseteq Y$ is open in Y if and only of $p^{-1}(U)$ is open in X.*

Remark. *An equivalent definition would be to have that p is a quotient map provided a subset $C \subseteq Y$ is closed in Y if and only if $p^{-1}(C)$ is open in X.*

Example 1.9.2. *Consider the mapping $p : X \to Y$ where if $U \subseteq X$ is open then $p(U) \subseteq Y$ is open. This is called an* ***open map****. It therefore follows from the definition that if p is a surjective continuous map that is open then it is a quotient map.*
The same applies for ***closed maps****.*

Example 1.9.3. *Let $X = [0,1] \cup [2,3], Y = [0,2]$ be subspaces of $\mathbb{R}$. Consider the map*

$$
p(x) = \begin{cases} x & \text{if } x \in [0,1] \\ x-1 & \text{if } x \in [2,3] \end{cases}
$$

It is easy to see that this function is surjective, continuous and closed. Hence it is a quotient map. It is not, however, an open map as the open set $[0, 1]$ *of* X *is not open in* Y.

Example 1.9.4. *Consider*

$$\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

a projection of $\mathbb{R}^2$ *onto the first coordinte. Indeed,* π_1 *is surjective and continuous. If* $U \times V$ *is a non-empty basis element for* $\mathbb{R} \times \mathbb{R}$ *then* $\pi_1(U \times V) = U$ *is open in* $\mathbb{R}$. *Hence* π_1 *is an open map and therefore a quotient map.*
π_1 *is not, however, a closed map. Consider the subset*

$$C = \{xy \,|\, xy = 1\}$$

of $\mathbb{R} \times \mathbb{R}$. *Indeed this subset is closed but* $\pi_1(C) = \mathbb{R} \backslash \{0\}$ *is not closed in* $\mathbb{R}$.

Definition 1.9.5. *Let* X *be a topological space and* $\sim$ *an equivalence relation on* X. *The* ***quotient space*** $X/\!\sim$ *on* X *is the set*

$$X/\sim = \{[x] \,|\, x \in X\} = \{\{v \in X \,|\, v \sim x\} \,|\, x \in X\}$$

Proposition 1.9.6. *Let* X *be a topological space,* $\sim$ *an equivalence relation on* X *and* $X/\!\sim$ *the corresponding quotient space of* X. *Consider the quotient map*

$$\begin{aligned} p : X &\to X/\sim \\ x &\mapsto [x] \end{aligned}$$

Now let

$$\tau_{X/\sim} = \{U \subseteq X/\sim \,|\, p^{-1}(U) \text{ is open in } X\}$$

then $\tau_{X/\sim}$ *is a topology on* $X/\!\sim$ *called the* ***quotient topology***.

Proof. We show that $\tau_{X/\sim}$ obeys the three axioms of a topology.

1. $\varnothing, X \in \tau_X$
   We have that $p^{-1}(\varnothing) = \varnothing$ and $p^{-1}(X/\sim) = X$. Hence $\varnothing, X/\sim \in \tau_{X/\sim}$.

2. $\forall\, I, \forall\, \{U_i\}_{i\in I} \in \tau_x, \bigcup_i U_i \in \tau_x$
   Let $\{U_i\}_{i\in I}$ be an arbitrary collection of unions of elements in X/$\sim$. We have that

$$p^{-1}\left(\bigcup_i U_i\right) = \bigcup_i p^{-1}(U_i)$$

   By assumption, $p^{-1}(U_i)$ is open in X and arbitrary unions of open sets of X are open hence $\bigcup_i p^{-1}(U_i)$ is open in X.

3. $U_1, U_2, \ldots, U_n \in \tau_x, \bigcap_i^n U_i \in \tau_x$
   Let $n \in N$ and $U_1, U_2, \ldots, U_n$ a finite collection of unions of elements in X/$\sim$. We have that

$$p^{-1}\left(\bigcap_i^n U_i\right) = \bigcap_i^n p^{-1}(U_i)$$

   By assumption, $p^{-1}(U_i)$ is open in X and finite intersections of open sets of X are open hence $\bigcap_i^n p^{-1}(U_i)$ is open in X.

$\square$

Remark. *It follows from the proof above that an equivalent definition of the quotient topology would be that a union of equivalence classes in X/$\sim$ is open if the union of their elements is open in X.*

Proposition 1.9.7. *Let X be a topological space, $\sim$ an equivalence relationship on X, and $p : X \to X/\sim$ an open quotient map. Then X/$\sim$ is Hausdorff if and only if $A = \{(x, y) \in X \times X \,|\, x \sim y\}$ is closed.*

Proof.

$\Longrightarrow$: Assume that X/$\sim$ is Hausdorff. Then $\forall\, [x], [y] \in$ X/$\sim, \exists\, U_x, U_y \subseteq$ X/$\sim$ open neighbourhoods of $[x], [y]$ respectively such that $U_{[x]} \cap U_{[y]} = \varnothing$. We want to show that A is closed. Hence it suffices to show that $(X \times X)\backslash A$ is open. Now,

$$(X \times X)\backslash A = \{(x, y) \in X \times X \,|\, x \nsim y\}$$

Since $p(x) = [x] \neq p(y) = [y]$, by hypothesis, we have two disjoint neighbourhoods of $p(x)$ and $p(y)$, namely $U_{[x]}$ and $U_{[y]}$ respectively.
Therefore, since p is a quotient map

$$\begin{aligned} &p^{-1}(U_{[x]} \cap U_{[y]}) = \varnothing \\ &\implies p^{-1}(U_{[x]}) \cap p^{-1}(U_{[y]}) = \varnothing \\ &\implies U_x \cap U_y = \varnothing \end{aligned}$$

where U_x and U_y are open neighbourhoods of x and y respectively. Hence around any point(x, y) we can excise an open neighbourhood, namely $U_x \times U_y$. Therefore $(X \times X)\backslash A$ is open.

$\Longleftarrow$: Now assume that A is closed. Hence its complement U is open. Since p is an open map, we have that

$$(p \times p)(U) := (p(x), p(y) \,|\, (x, y) \in U)$$

is open in (X/∼)×(X/∼).
Now, the complement $V =$ (X/∼)×(X/∼)$\backslash(p \times p)(U) = \{([y], [y]) \,|\, [y] \in X/\sim\}$ is closed.
Consider two equivalence classes $[x] \neq [y] \in X/\sim$. Then $([x], [y]) \notin V$. Hence there is an open neighbourhood of $([x], [y])$ that does not intersect V. Such a neighourhood is of the form $U_x \times U_y$ where $U_x \cap U_y = \varnothing$. Hence X/∼ is Hausdorff.

□

Chapter 2

Connectedness and Compactness

2.1 Connected Spaces

Definition 2.1.1. *Let X be a topological space. A **seperation** of X is a pair U, V of disjoint non-empty open subsets of X where $U \cup V = X$. X is called **connected** if there does not exist a seperation of X.*

Remark. *If $X = U \cup V$ is a seperation then both U and V are clopen. We can therefore reformulate the definition of connectedness to be that X is connected if and only if there are no non-trivial clopen subsets.*

Example 2.1.2. *Consider X equipped with the discrete topology. Then X is connected if and only if $|X| = 1$.*

Example 2.1.3. *Consider X equipped with the discrete topology. Then X is connected.*

Example 2.1.4. *Consider $A \subseteq \mathbb{R}$ any open or closed subset. Then A is connected as we can never find two non-empty open sets that seperate A.*

Example 2.1.5. *Consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$ and $a, b \in \mathbb{R} \backslash \mathbb{Q}$. We have that $(a, b) \cap \mathbb{Q}$ is an open set in the subspace topology. We also have that $[a, b] \cap \mathbb{Q}$ is a closed set in the subspace topology. However, $U = (a, b) \cap \mathbb{Q} = [a, b] \cap \mathbb{Q}$. U is therefore a clopen set of $\mathbb{Q}$ and hence $\mathbb{Q}$ is not connected.*

Example 2.1.6. *Let X be a topological space and $Y \subseteq X$ a subspace.*

Lemma 2.1.7. *Let X be a topological space and $Y \subseteq X$ a subspace. A seperation of Y is a pair of disjoint non-empty sets A and B whose union is Y, neither of which contains a limit point of the other.*

Proof.

$\Longrightarrow:$ Suppose that A and B form a seperation of Y. We want to show that A and B contain no limit points of each other. It follows from the definition of a seperation that A is a clopen set in Y. The closure of $A = \overline{A} \cap Y$ (where $\overline{A}$ is the closure of A in X). Now, since A is closed in Y, we have that

$$\begin{aligned} & A = \overline{A} \cap Y \\ & \Longrightarrow \varnothing = A \cap B = \overline{A} \cap Y \cap B \\ & \Longrightarrow \overline{A} \cap B = \varnothing \\ & \Longrightarrow (A' \cup A) \cap B = \varnothing \\ & \Longrightarrow (A' \cap B) \cup (A \cap B) = \varnothing \\ & \Longrightarrow A' \cap B = \varnothing \end{aligned}$$

Hence B contains no limit points of A. A similar argument can be applied to B.

$\Longleftarrow:$ Now assume that A and B are disjoint non-empty sets whose union is Y, neither of which contains a limit point of the other. We have that $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$. Hence $\overline{A} \cap Y = A$ and $\overline{B} \cap Y = B$. This implies that A and B are closed in Y and since $A = Y \backslash B$ and $B = Y \backslash A$, they are both open as well.

□

Lemma 2.1.8. *Let X be a topological space, $Y \subseteq X$ a subspace and C and D a seperation of X. If Y is connected then it lies entirely within C or D.*

Proof. Suppose that $Y \cap C \neq \varnothing$ and $Y \cap D \neq \varnothing$. Since C and D are both open in X, by the definition of the subspace topology, they are both open in Y. Since we have that $(Y \cap C) \cup (Y \cap D) = Y$ we have that $Y \cap C$ and $Y \cap D$ form a seperation of Y. This contradicts the fact that Y is connected and hence either $Y \cap C = \varnothing$ or $Y \cap D = \varnothing$. Therefore, Y lies entirely within C or D. □

Theorem 2.1.9. *Let X be a topological space. The union of a collection of connected subspaces of X that have a point in common is connected.*

Proof. Let $\{U_i\}_{i \in I}$ be a collection of connected subspaces of X such that $\bigcap_{i \in I} U_i = p$ for some point p. We want to show that there exists no seperation of $U = \bigcup_{i \in I} U_i$.
Suppose A and B form a seperation of U. We have that $p \in A$ or $p \in B$. Suppose, without loss of generality, that $p \in A$. Since $p \in A$ and $\forall\, i, p \in U_i$, we have that $\forall\, i, U_i \subseteq A$. If not then $\exists\, x \in U_i$ such that $x \in B$, contradicting the fact that U_i is connected.
Since $U = \bigcup_{i \in I} U_i$ and $\forall\, i, U_i \subseteq A$ we then have that $B = \varnothing$. This contradicts the fact that B is non-empty hence U must be connected. □

Theorem 2.1.10. *Let X be a topological space and $Y \subseteq X$ a subspace. If $A \subseteq B \subseteq \overline{A}$ then B is also connected.*

Proof. Suppose X and Y form a seperation of B. Then, since A is connected, $A \subseteq X$ or $A \subseteq Y$. Suppose, without loss of generality, that $A \subseteq X$. Then, $\overline{A} \subseteq \overline{X}$. Hence we have that $B \subseteq \overline{A} \subseteq \overline{X}$. Since $\overline{X}$ and Y are disjoint, B cannot intersect Y which contradicts the fact that Y is a non-empty subset of B. Hence B is connected. □

Theorem 2.1.11. *Let X and Y be topological spaces and $f : X \to Y$ a continuous map. If X is connected then $f(X) \subseteq Y$ is also connected.*

Proof. We want to show that the image space Z = f(X) is connected. If we restrict the range to Z then the map is still continuous and is surjective. Hence we consider $g : X \to Z$. Suppose C and D form a seperation of Z. Then by definition $C \cup D = Z, C \cap D = \varnothing$ and C and D both non-empty.
Since f is a continuous map, we have that $g^{-1}(C)$ and $g^{-1}(D)$ are both open in X.
Now,

$$\begin{aligned} &g^{-1}(C \cup D) = g^{-1}(Z) \\ &\implies g^{-1}(C) \cup g^{-1}(D) = X \end{aligned}$$

$$\begin{aligned} &g^{-1}(C \cap D) = g^{-1}(\varnothing) \\ &\implies g^{-1}(C) \cap g^{-1}(D) = \varnothing \end{aligned}$$

We also have that since g is surjective, $g^{-1}(C)$ and $g^{-1}(D)$ are non-empty.
Hence $g^{-1}(C)$ and $g^{-1}(D)$ form a seperation of X, contradicting the fact that X is connected. Therefore, the image of X under a continuous map is connected. □

Theorem 2.1.12. *Let* $\{X_i\}_{1\leq i\leq n}$ *be a finite collection of connected spaces. Then the finite cartesian product* $X = \prod_{i=1}^{n} X_i$ *is also connected.*

Proof. We shall prove the theorem using induction. For the basis case, we consider the product of two connected spaces X and Y.
Consider the point $(a, b) \in X \times Y$. The horizontal slice $X \times \{b\}$ is connected as it is homeomorphic to X. The vertical slices $\{x\} \times Y$ are also connected as they are homeomorphic to Y.
Hence each T-shaped space

$$T_x = (X \times \{b\}) \cup (\{x\} \times Y)$$

is connected as it is the union of two connected spaces that have the point (x, b) in common.
Now consider

$$T = \bigcup_{x \in X} T_x$$

of all such T-shaped spaces. This is also a connected space as it is the union of connected spaces that all contain the point (a, b) in common. Since $T = X \times Y$, we have that $X \times Y$ is a connected space.
Now assume that the theorem is true for $n = k$. We prove that it is true for $n = k + 1$.
We have that

$$X_1 \times \cdots \times X_{n_1} = (X_1 \times \cdots \times X_n) \times X_{n+1}$$

By assumption, $X_1 \times \cdots \times X_n)$ and X_{n+1} are connected. By the basis case, their product is connected as the product of two connected spaces. Hence any finite cartesian product of connected spaces is connected. □

Proposition 2.1.13. *Consider* $X = \mathbb{R}^\omega$ *equipped with the box topology. Then* X *is not a connected space.*

Proof. We show that $\mathbb{R}^\omega$ has a seperation consisting of bounded and unbounded sequences of real numbers.
Let $U \subseteq \mathbb{R}^\omega$ be the subset of $\mathbb{R}^\omega$ consisting of bounded sequences of real numbers. Then given any $x \in \mathbb{R}^\omega$ bounded sequence, we have that

$$\prod_i (x_1 - 1, x_1 + 1) \times (x_2 - 1, x_2 + 1) \times \ldots$$

is an open neighbourhood for x. Hence U is an open set in $\mathbb{R}^\omega$.
Now consider $V \subseteq \mathbb{R}^\omega$ the set of unbounded sequences of real numbers. Given any $x \in \mathbb{R}^\omega$, we have that

$$\prod_i (x_1 - 1, x_1 + 1) \times (x_2 - 1, x_2 + 1) \times \dots$$

is also an open neighbourhood of x. Hence V is also open in $\mathbb{R}^\omega$.
Since U and V are obviously non-empty and disjoint and also open, it follows that they form a seperation of $\mathbb{R}^\omega$ in the box topology. □

Proposition 2.1.14. *Consider $X = \mathbb{R}^\omega$ equipped with the product topology. Then X is a connected space.*

Proof. Let $\tilde{\mathbb{R}}^n$ denote the subspace of $\mathbb{R}^\omega$ consisting of sequences $x = (x_1, x_2, \dots)$ such that $x_i = 0$ for $i > n$. Then $\tilde{\mathbb{R}}^n$ is clearly homeomorphic to $\mathbb{R}^n$ and hence connected. Now denote $\mathbb{R}^\infty = \bigcup_{i=1}^{\infty} \tilde{\mathbb{R}}^n$. This set is also connected as it is the union of connected spaces all with the point $(0, 0, \dots)$ in common. It now suffices to show that $\mathbb{R}^\omega$ is the closure of $\mathbb{R}^\infty$ and thus by a previous theorem, $\mathbb{R}^\omega$ is connected.
Let $a = (a_1, a_2, \dots) \in \mathbb{R}^\omega$. We show that every basic neighbourhood of a intersects $\mathbb{R}^\infty$. Let

$$U = \prod_{1 \leq i \leq k} U_i \times \prod_{i \geq k} \mathbb{R}$$

be a basic neighbourhood of a where each U_i is a basic open neighbourhood of a_i.
Consider the point

$$x = (a_1, \dots, a_n, 0, 0, \dots)$$

of $\mathbb{R}^\infty$. Then $x \in U$ since $a_i \in U_i \; \forall \, i$ and $0 \in \mathbb{R} \; \forall \, i \geq k$. Hence $\mathbb{R}^\omega$ is the closure of $\mathbb{R}^\infty$ and is thus closed. □

2.2 Connected Subspaces of the Real Line

Theorem 2.2.1. *(Intermediate Value Theorem) Let X be a connected topological space and $f : X \to \mathbb{R}$ a continuous map. Consider $a, b \in X$ and $r \in \mathbb{R}$ such that $f(a) \leq r \leq f(b)$. Then there exists a point $c \in X$ such that $f(c) = r$.*

Proof. Assume that there is no point $c \in X$ such that $f(c) = r$. Now consider the two sets

$$A = f(X) \cap (\infty, r)$$
$$B = f(X) \cap (r, \infty)$$

They are obviously disjoint and they are non-empty as $f(a) \subseteq A$ and $f(b) \subseteq B$. We can also see that A and B are open in $f(X)$ being the intersection of $f(X)$ with an open set of $\mathbb{R}$. Since there is no point in X with r its image, we see that $f(X) = A \cup B$. Hence A and B form a seperation of $f(X)$. However X is connected and this contradicts the fact that the image of a connected set under a continuous map is connected. Hence there must be a point $c \in X$ such that $f(c) = r$. □

Definition 2.2.2. *Let X be a space and $x, y \in X$. A* ***path from x to y*** *is a continuous map $f : [a, b] \to X$ where $a, b \in \mathbb{R}$ such that $f(a) = x$ and $f(b) = y$. A space X is said to be* ***path connected*** *if every pair of points of X can be joined by a path in X.*

Theorem 2.2.3. *Let X be a topolgical space. If X is path connected then it is connected.*

Proof. Suppose that A and B form a seperation of X. Let $f : [a, b] \to X$ be a path in X. Since $[a, b]$ is connected and f is a continuous map, it follows that $f([a, b])$ is also connected. Hence $f([a, b])$ lies entirely within A and B. There is therefore no path connecting a point in A to a point in B, contradicting the fact that X is path connected. Hence X must be connected. □

Example 2.2.4. *Any open or closed ball in $\mathbb{R}^n$ is path connected.*

Proposition 2.2.5. *Consider the set*

$$S = \left\{ (x, sin\left(\frac{1}{x}\right) \middle| 0 < x \leq 1 \right\} \cup (\{0\} \times [-1, 1])$$

Then S is connected but not path connected. It is called the ***topologist's sine curve****.*

Proof. S is a connected subset of $\mathbb{R}^2$. We want to show that S is not path-connected.
Assume that there exists a path $f : [0, 1] \to S$ from the origin to another

point in S. Let $f(t) = (x(t), y(t))$. Then $x(0) = 0$ and $x(t) > 0, y(t) = sin\left(\frac{1}{x}\right)$ for $t > 0$.
We shall show that there exists a sequence of points $t_n \to 0$ such that $y(t_n) = (-1)^n$. Then $y(t_n)$ doesn't converge, contradicting the continuity of f.
Let $n > 0$ and choose u such that $0 < u < x\left(\frac{1}{n}\right)$ and $sin\left(\frac{1}{n}\right) = (-1)^n$. We can then apply the intermediate value theorem to find t_n such that $0 < t_n < \frac{1}{n}$ and $x(t_n) = u$. □

2.3 Connected Components

Definition 2.3.1. *Let X be a topological space and define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the* ***connected components*** *of X.*

Theorem 2.3.2. *Let X be a topological space. The connected components of X are connected disjoint subspaces of X whose union is X. Each non-empty connected subspace of X intersects only one connected component of X.*

Proof. Since the components are simply equivalence classes, it follows that they are disjoint and their union is X.
Now assume that there exists a connected subspace $A \subseteq X$ that intersects two connected components C_1 and C_2. Let $x \in C_1$ and $y \in C_2$ be two such intersections. By definition of, $x \sim y$ hence they are in the same connected component. This is obviously a contradiction hence each connected subspace must intersect at most one connected component. It remains to show that each connected component is indeed connected.
Choose $x_0 \in C$. $\forall x \in C, x_0 \sim x$ hence there is a connected subspace A_x containing both x_0 and x. By the result just proved, we have that $A_x \subset C$. Hence

$$C = \bigcup_{x \in C} A_x$$

Since the each A_x is connected and they each share the point x_0 in common, their union is connected. □

Definition 2.3.3. *Let X be at topological space and define an equivalence relation on X by setting $x \sim y$ if there is a path in X from x to y. The equivalence classes are called the* ***connected components*** *of X.*

Theorem 2.3.4. *Let X be a topological space. The path connected components of X are path-connected disjoing subspaces of X whose union is X. Each non-empty subspace of X intersects at most one path-connected component.*

Proof. The proof is left as an exercise to the reader. It follows the same argumentation as the previous proof. □

Example 2.3.5. *Consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$. The connected components of $\mathbb{Q}$ are single points.*

Example 2.3.6. *Consider the topologist's sine curve. This set has one connected component (the whole set) and two path connected components: $\{0\} \times [-1, 1]$ and $\left\{(x, sin\left(\frac{1}{x}\right) \,\middle|\, 0 < x \leq 1\right\}$.*

2.4 Compact Spaces

Definition 2.4.1. *Let X be a space. A collection $\mathcal{C}$ of subsets of X is said to **cover** X or be a **covering** of X if the union of elements in $\mathcal{A}$ is equal to X. $\mathcal{A}$ is said to be an **open covering** of X if it consists of open subsets of X.*

Definition 2.4.2. *A topological space X is said to be **compact** if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.*

Example 2.4.3. *The real line $\mathbb{R}$ is not compact. Consider the open covering of $\mathbb{R}$*

$$\mathcal{A} = \{(n, n+2) \,|\, n \in \mathbb{Z}\}$$

This covering contains no finite subcollection that covers $\mathbb{R}$.

Example 2.4.4. *Consider the following subsoace of $\mathbb{R}$:*

$$X = \{0\} \cup \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{Z}_+ \right\}$$

Given an open covering $\mathcal{A}$ of X, there is an element $U \in \mathcal{A}$ containing 0. Since U is open, it contains all but finitely many of the points $\frac{1}{n}$. For each such point, choose an element of $\mathcal{A}$ that contains in. The collection consisting of these elements of $\mathcal{A}$ along with U is a finite subcollection of $\mathcal{A}$ that covers X. Hence this subset is compact.

Example 2.4.5. *Any finite topological space X is necessarily compact as every open covering of X is finite.*

Example 2.4.6. *Consider the interval $(0,1)$ and the open covering*

$$\mathcal{A} = \left\{ \left(\frac{1}{n}, 1 \right) \middle| \, n \in \mathbb{Z}_+ \right\}$$

This covering contains no finite subcollection covering $(0,1)$ hence $(0,1)$ is not compact.

Definition 2.4.7. *Let X be a topological spoce and $Y \subseteq X$ a subspace. Then a collection $\mathcal{A}$ of subsets of X is said to* ***cover Y*** *if the union of its elements contains Y.*

Lemma 2.4.8. *Let X be a topological space and $Y \subseteq X$ a subspace. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.*

Proof.

$\Longrightarrow$: Assume that Y is compact. Consider the open covering $\mathcal{A} = \{A_i\}_{i \in I}$ of Y by sets open in X. Then the collection $\{A_i \cap Y \mid i \in I\}$ is a covering of Y by sets open in Y. Hence a finite subcollection

$$\{A_{i_1} \cap Y, A_{i_2} \cap Y, \ldots, A_{i_n} \cap Y\}$$

is a covering of Y by sets open in Y. This implies that

$$\{A_{i_1}, A_{i_2}, \ldots, A_{i_n}\}$$

is a subcollection of $\mathcal{A}$ that covers Y.

$\Longleftarrow$: Now suppose that every covering of Y by sets open in X contains a finite subcollection covering Y. We want to show that Y is compact. Let $\mathcal{A}' = \{\mathcal{A}_i\}$ be a covering of Y by sets open in Y. For each i, choose a set $\mathcal{A}_i$ open in X such that $\mathcal{A}'_i = \mathcal{A}_i \cap Y$.
The collection $\mathcal{A} = \{\mathcal{A}_i\}$ is a covering of Y by sets open in X. By assumption, some finite subcollection $\{\mathcal{A}_{i_1}, \ldots, \mathcal{A}_{i_n}\}$ covers Y.
Then $\{\mathcal{A}'_{i_1}, \ldots, \mathcal{A}'_{i_n}\}$ is a finite subcollection of $\mathcal{A}'$ that covers Y.
Hence Y is compact.

□

Theorem 2.4.9. *Let X be a compact topological space. Then every closed subspace of is compact.*

Proof. Let X be a compact topological space and $Y \subseteq X$ a closed subspace. Given a covering $\mathcal{A}$ of Y by sets open in X, we can form an open covering $\mathcal{B}$ of X by adjoining the open set $X \backslash Y$ to $\mathcal{A}$:

$$\mathcal{B} = \mathcal{A} \cup (X \backslash Y)$$

Since X is a compact space, some finite subcollection of $\mathcal{B}$ covers X. If the subcollection contains $X \backslash Y$, discard $X \backslash Y$. The resulting collection is a finite collection of $\mathcal{A}$ that covers Y. □

Theorem 2.4.10. *Let X be a Hausdorff space. Then every compact subspace of X is closed.*

Proof. Let X be a Hausdorff space and consider a compact subspace $Y \subseteq X$. We want to show that Y is closed. This is equivalent to proving that $X \backslash Y$ is open. It is therefore sufficient to show that given a point $x_o \in X \backslash Y$, every neighbourhood of x_0 is disjoint from Y.
Given a point $y \in Y$, consider the disjoint neighbourhoods U_y and V_y of x_0 and y, the existence of which is guaranteed by the fact that X is Hausdorff. The collection $\{V_y \,|\, y \in Y\}$ is a covering of Y by sets open in X. Therefore, finitely many of them $\{V_{y_1}, \ldots, V_{y_n}\}$ cover Y.
The open set

$$V = V_{y_1} \cup \cdots \cup V_{y_n}$$

contains Y and is disjoint from the open set

$$U = U_{y_1} \cap \cdots \cap U_{y_n}$$

where each U_{y_i} is the corresponding neighbourhood of x_o. Indeed if $z \in V$ then $z \in V_{y_i}$ for some i, hence $z \notin U_{y_i} \implies z \notin U$. Hence U is a neighbourhood of x_0 disjoint from Y. Thus Y is closed. □

Theorem 2.4.11. *The image of a compact space under a continuous map is compact.*

Proof. Let $f : X \to Y$ be a continuous map where X is a compact topological space. We want to show that for every open covering of $f(X) \subseteq Y$, there

exists a finite subcollection that covers $f(X)$.
Let $\mathcal{A}$ be a covering of the $f(X)$ by sets open in Y. The collection

$$\{f^{-1}(A) \mid A \in \mathcal{A}\}$$

is a collection of sets covering X. They are open as f is continuous. Hence finitely many of them, say $f^{-1}(A_1), \ldots, f^{-1}(A_n)$ cover X. Hence the sets $A_1, \ldots, A_n$ cover $f(X)$. □

Theorem 2.4.12. *Let X and Y be two topological spaces and $f : X \to Y$ a bijective continuous function. If X is compact and Y is Hausdorff then f is a homeomorphism.*

Proof. It is sufficient to show that the image of a closed set of X under f is closed in Y hence showing that f^{-1} is continuous. If A is closed in X then A is compact by a previous theorem. Hence by the previous theorem, $f(A)$ is also compact. By another theorem, we see that since Y is Hausdorff, $f(A)$ is closed in Y. □

Lemma 2.4.13. *(Tube Lemma)*
Consider the product space $X \times Y$ where Y is compact. If N is an open set of $X \times Y$ containing the slice $\{x_0\} \times Y$ of $X \times Y$ then N contains some tube $W \times Y$ about $\{x_0\} \times Y$ where W is a neighbourhood of $x_o \in X$.

Proof. Let X be a topological space and Y a compact space, $x_0 \in X$ and $N \subseteq X \times Y$ an open set containing the slice $\{x_0\} \times Y$.
Since the slice $\{x_0\} \times Y$ is homeomorphic to Y and Y is compact, it follows that $\{x_0\} \times Y$ is also compact. We can therefore cover $\{x_0\} \times Y$ with finitely many basis elements, say

$$U_1 \times V_1, \ldots, U_n \times V_n$$

Now define

$$W = U_1 \cap \cdots \cap U_n$$

The set W is open as it is the finite intersection of open sets and it contains x_0.
We show that the sets $U_i \times V_i$ actually cover $W \times Y$.
Let $(x, y) \in W \times Y$. Consider the point $(x_0, y) \in \{x_0\} \times Y$ having the same y-coordinate. $(x_0, y) \in U_i \times V_i$ for some i. Hence $y \in V_i$. But $x \in U_i \ \forall i$ thus $(x, y) \in U_i \times V_i$. Since all the sets $U_i \times V_i$ lie in N and since they cover $W \times Y$, the tube also lies in N. □

Theorem 2.4.14. *The product of finitely many compact spaces is compact.*

Proof. We shall prove the theorem using induction. For the basis case, we prove the theorem for the product of two spaces.
Let $\mathcal{A}$ be an open covering of $X \times Y$. Given $x_0 \in X$, the slice $\{x_0\} \times Y$ is compact as it is homeomorphic to the compact space Y. It can therefore be covered by finitely many elements $A_1, \ldots, A_m$ of $\mathcal{A}$. Their union $N = A_1 \cup \cdots \cup A_m$ is an open set containing $\{x_0\} \times Y$.
By the Tube Lemma, the open set A contains a tube $W \times Y$ about $\{x_0\} \times Y$ where W is open in X. Then $W \times Y$ is covered by finitely many elements $A_1, \ldots, A_m$ of $\mathcal{A}$.
Hence $\forall\, x \in X$, we can choose a neighbourhood W_x of x such that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$. The collection of all such W_x is an open covering of X. By compactness of X, there exists a finite subcollection $W_1, \ldots, W_k$ covering X. The union of the tubes $W_1 \times Y, \ldots, W_k \times Y$ is all of $X \times Y$. Since each one can be covered by finitely many elements, ther union (and hence $X \times Y$) can also be covered by finitely many elements.
Now assume that the theorem is true for $X_1 \times \cdots \times X_n$. We want to show that it holds for $X_1 \times \cdots \times X_{n+1}$. $X_1 \times \cdots \times X_{n+1}$ is homeomorphic to $(X_1 \times \cdots \times X_n) \times X_{n+1}$. Since this is the product of two compact spaces $X_1 \times \cdots \times X_n$ and X_{n+1}, it follows by the basis case that their product is also compact. Hence by induction, the product of finitely many compact spaces is compact. □

Definition 2.4.15. *A collection $\mathbb{C}$ of subsets of a space X is said to have the* ***finite intersection property*** *if for every finite subcollection $\{C_1, \ldots, C_n\}$, their intersection is nonempty.*

Remark. *Another definition of compactness is if for every collection $\mathcal{C}$ of closed sets in X that has the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}}$ is non-empty. This follows from the properties of open and closed sets and the De Morgan Law.*

2.5 Compact Subspaces of the Real Line

Theorem 2.5.1. *A subspace A of $\mathbb{R}^n$ is compact if and only if it is bounded in the Euclidian metric d or the square metric ρ.*

Proof. It suffices to consider only the metric ρ as the inequalities

$$\rho(x,y) \leq d(x,y) \leq \sqrt{n}\rho(x,y)$$

$\Longrightarrow$: Suppose that A is compact. Then by Theorem 2.4.10, A is closed. Now consider the collection of open sets

$$\mathcal{B} = \{B_\rho(0,m) \,|\, m \in \mathbb{Z}_+\}$$

whose union is all of $\mathbb{R}^n$. Since A is compact, some finite subcollection of $\mathcal{B}$ must cover A. It follows that $A \subseteq B_\rho(0,M)$ for some M. Therefore, for any two points x and y of A we have that $\rho(x,y) \leq 2M$. Thus, A is bounded under ρ.

$\Longleftarrow$: Now assume that A is closed and bounded under ρ. Suppose that $\rho(x,y) \leq N$ for every pair x, y of points of A. Choose a point $x_0 \in A$ and let $\rho(x_0,0) = b$. The triangle inequality implies that $\rho(x,0) \leq N + b \,\forall\, x \in A$. If $P = N + b$ then A is a subset of the cube $[-P,P]^n$ which is compact. Since A is closed, it follows that it is also compact.

□

Theorem 2.5.2. *(Extreme Value Theorem)*
Let X be a topological space and $f : X \to \mathbb{R}$ a continuous function. If X is compact then there exists points $c, d \in X$ such that $f(c) \leq f(x) \leq f(d) \,\forall\, x \in X$.

Proof. Since f is a continuous map and X is compact, it follows that $A = f(X) \subseteq \mathbb{R}$ is a compact subset. We will show that A has a maximal element M and a minimal element m. We have that $m = f(c)$ and $M = f(d)$ for some points $c, d \in X$.
Assume that A has no maximal element. Then the collection

$$\{(-\infty, a) \,|\, a \in A\}$$

forms an opening covering of A. Since A is compact, some finite subcollection

$$\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$$

covers A. If a_i is the largest of the elements $a_1, \ldots, a_n$ then a_i nellongs to none of these sets, contrary to the fact that it forms part of a finite subcovering of A. Hence A must have a maximal element. A similar argument can be applied to show that A must have a minimal element. □

Definition 2.5.3. *Let (X, d) be a metric space and $A \subseteq X$ a subset. For all $x \in X$, we define the* ***distance from x to A*** *by the equation*

$$d(x, A) = inf d(x, a) \,|\, a \in A$$

Definition 2.5.4. *Let (X, d) be a metric space and $A \subseteq X$ be a bounded subset. We define the* ***diametre*** *of A to be*

$$diam(A) = sup\{d(a_1, a_2) \,|\, a_1, a_2 \in A\}$$

Lemma 2.5.5. *(Lebesgue number lemma)*
Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diametre less than δ, there exists an element of $\mathcal{A}$ covering it. The number δ is called a ***Lebesgue number*** *for the covering $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be an open covering of X. If X is an element of $\mathcal{A}$ then any positive nunber is a Lebesgue number for $\mathcal{A}$ and we are done. Hence assume X is not an element of $\mathcal{A}$.
Choose a finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ that covers X. For each i, set $C_i = X \backslash A_i$. Now define $f : X \to \mathbb{R}$ by letting $f(x)$ be the average of the numbers $d(x, C_i)$:

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i)$$

First we show that $f(x) > 0 \; \forall x \in X$. Given $x \in X$, choose i such that $x \in A_i$. Now choose ε so that the ε-neighbourhood of x lies in A_i. Then $d(x, C_i) \geq \varepsilon$ so that $f(x) \geq \frac{\varepsilon}{n}$.
Since f is continuous, it must have a minimum value δ. We shall show that δ is indeed the Lebesgue number for $\mathcal{A}$. Let B be a subset of X of diametre less than δ. Now choose a point $x_0 \in B$. Then B lies in the δ-neighbourhood of x_0. We have that

$$\delta \leq f(x_0) \leq d(x_0, C_m)$$

where $d(x_0, C_m)$ is the largest of the numbers $d(x_0, C_i)$. Then the δ-neighbourhood of x_0 is contained in the element $A_m = X \backslash C_m$ of the covering $\mathcal{A}$. □

Definition 2.5.6. *A function fron the metric space (X, d_X) to the metric space (Y, d_Y) is said to be* ***uniformly continuous*** *if given $\varepsilon > 0$, there is a $\delta > 0$ such that for every pair of points $x_0, x_1 \in X$,*

$$d_X(x_0, x_1) < \delta \implies d_Y(f(x_0), f(x_1)) < \varepsilon$$

Theorem 2.5.7. *(Uniform Continuity Theorem)*
Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.

Proof. Let $\varepsilon > 0$ and take the open covering of Y by balls $B(y, \frac{\varepsilon}{2})$. Let $\mathcal{A}$ be an open covering of X given by the inverse images of these balls under f. Now choose δ to be a Lebesgue number for the covering $\mathcal{A}$. Then if $x_1, x_2 \in X$ are two points such that $d_X(x_1, x_2) < \delta$, the two point set $\{x_1, x_2\}$ has diameter less than δ so that its image $\{f(x_1), f(x_2)\}$ lies in some ball $B(y, \frac{\varepsilon}{2})$. Then $d_Y(f(x_1), f(x_2)) < \varepsilon$. Hence f is uniformly continuous. □

2.6 Local Compactness

Definition 2.6.1. *Let X be a topological space. We say that X is* ***locally compact at x*** *if there is some compact subspace C of X that contains a neighbourhood of x. If X is locally compact at all of its points, X is said to be* ***locally compact****.*

Example 2.6.2. *The real line $\mathbb{R}$ is locally compact. Indeed given any point $x \in \mathbb{R}$ we can take the open neighbourhood $(x-1, x+1)$ which lies in the compact subspace $[x-1, x+1]$.*

Example 2.6.3. *The rationals $\mathbb{Q}$ as a subspace of $\mathbb{R}$ are not locally compact. Indeed any compact subspace of $\mathbb{Q}$ is necessarily a single point. We cannot squeeze an open neighbourhood of $\mathbb{Q}$ around a point inside any of these compact subspaces.*

Example 2.6.4. *The space $\mathbb{R}^n$ is locally compact. Indeed, consider a point $x \in \mathbb{R}^n$. Then $x \in (x_1 - 1, x_1 + 1) \times \cdots \times (x_n - 1, x_n + 1)$ which is contained in the compact subspace $[x_1 - 1, x_1 + 1] \times \cdots \times [x_n - 1, x_n + 1]$.*

Example 2.6.5. *The space $\mathbb{R}^w$ is not locally compact. Take a point $x \in \mathbb{R}^w$. Then $x \in (x_1 - 1, x_1 + 1) \times \cdots \times (x_n - 1, x_n + 1) \times \prod_{i>n} \mathbb{R}$. If this open* *neighbourhood were contained in a compact subspace, then its closure*

$$[x_1 - 1, x_1 + 1] \times \ldots [x_n - 1, x_n + 1] \times \prod_{i>n} \mathbb{R}$$

would be compact which it is not.