Tate Cohomology

Based off *Class Field Theory - The Bonn Lectures* by Jürgen Neukirch

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Contents

1 G-modules

Throughout this section, G shall be a finite group written multiplicatively.

1.1 Definitions

Definition 1.1.1. Let A be an abelian group. We say that A is a ***G*-module** if there exists a function $\rho : G \times A \to A$ such that for all $\sigma, \tau \in G$ and $a, b \in A$ we have

1. $\rho(1, a) = a$
2. $\rho(\sigma, a + b) = \rho(\sigma, a) + \rho(\sigma, b)$
3. $\rho(\sigma\tau, a) = \rho(\sigma, \rho(\tau, a))$

Such a function is referred to as a ***G*-action** and we shall simply write a^σ for $\rho(\sigma, a)$. Morevoer, we write A^G for the subgroup of A left fixed by the action of G.

Definition 1.1.2. Let A and B be G-modules. We say that a homomorphism of groups $\phi : A \to B$ is a ***G*-homomorphism** if it commutes with the action of G: for all $a \in A$ and $\sigma \in G$ we have $\phi(a)^\sigma = \phi(a^\sigma)$.

Definition 1.1.3. We define the **category of *G*-modules**, denoted $\boldsymbol{G}_{\mathbf{mod}}$, to be the one with objects the G-modules and morphisms the G-homomorphisms.

Proposition 1.1.4. *Let A be a G-module and H a subgroup of G. Then*

1. *A is an H-module.*
2. *If H is normal in G then A^H is a G/H-module.*

Proof.

Part 1: This is immediate upon realising the action of H on A is given by the restriction of the action of G on A to the subgroup H.

Part 2: We first define the action of G/H on A^H as follows. Given $a \in A^H$ and $[\sigma] \in G$, define $a^{[\sigma]}$ to be a^σ. The fact that this satisfies the axioms of a G/H-action is immediate by construction so it suffices to show that this action is indeed well-defined. To this end, suppose that $[\sigma] = [\tau]$ for some $\sigma, \tau \in G$. By definition, $\tau = \sigma\chi$ for some $\chi \in H$. Then

$$a^{[\tau]} = a^{[\sigma\chi]} = a^{\sigma\chi} = (a^\chi)^\sigma = a^\sigma = a^{[\sigma]}$$

□

1.2 Group Rings

Definition 1.2.1. Let R be a commutative ring. We define the **group ring of *G* over *R***, denoted $R[G]$, to be the free R-module on G. In other words,

$$R[G] = \left\{ \sum_{\sigma \in G} r_\sigma \sigma \;\middle|\; r_\sigma \in R \right\}$$

Proposition 1.2.2. *The category $\boldsymbol{G}_{\mathbf{mod}}$ is isomorphic to the category $\mathbf{Mod}_{\mathbb{Z}[G]}$ of $\mathbb{Z}[G]$-modules.*

Proof. It suffices to exhibit a functor $F : \boldsymbol{G}_{\mathbf{mod}} \to \mathbf{Mod}_{\mathbb{Z}[G]}$ with an inverse. To this end, fix G-modules A, B and a G-homomorphism $\varphi : A \to B$. Define FA to be the $\mathbb{Z}[G]$-module with $\mathbb{Z}[G]$-multiplication given by

$$\left(\sum_{\sigma\in G} n_\sigma \sigma\right) \cdot a = \sum_{\sigma\in G} n_\sigma a^\sigma$$

for $\sum_{\sigma\in G} n_\sigma\sigma \in \mathbb{Z}[G]$ and $a \in A$. Define $F(A \xrightarrow{\phi} B)$ to be exactly ϕ as a homomorphism of abelian groups. Then the defining property of a G-homomorphism induces the structure of a $\mathbb{Z}[G]$-module homomorphism on ϕ.

To see that F has an inverse, we define $F^{-1} : \mathbf{Mod}_{\mathbb{Z}[G]} \to \boldsymbol{G}_{\mathbf{mod}}$ as follows. Fix a $\mathbb{Z}[G]$-module M. We can easily make M into a G-module as follows: given $\sigma \in G$ and $m \in M$, let $m^\sigma = \sigma \cdot m$ where the latter is the $\mathbb{Z}[G]$-module multiplication of $\sigma \in \mathbb{Z}[G]$ with m. Let $F^{-1}M$ be this G-module. Given a $\mathbb{Z}[G]$-module homomorphism $\varphi : M \to N$, let $F^{-1}\varphi$ be the induced homomorphism of G-modules. It is guaranteed to to be a G-module by the defining properties of a $\mathbb{Z}[G]$-module homomorphism. □

Definition 1.2.3. Let A be a G-module. We say that A is **$\mathbb{Z}[G]$-free** (or simply **G-free**) if A admits a decomposition into a direct sum of G-submodules of A that are all isomorphic to $\mathbb{Z}[G]$. In other words, we can write

$$A = \bigoplus_{i\in I} \mathbb{Z}[G]$$

for some indexing set I.

Definition 1.2.4. We define the **augmentation** of $\mathbb{Z}[G]$ to be the homomorphism

$$\begin{aligned} \varepsilon : \mathbb{Z}[G] &\to \mathbb{Z} \\ \sum_{\sigma\in G} n_\sigma\sigma &\mapsto \sum_{\sigma\in G} n_\sigma \end{aligned}$$

Its kernel

$$I_G = \left\{ \sum_{\sigma\in G} n_\sigma\sigma \;\middle|\; \sum_{\sigma\in G} n_\sigma = 0 \right\}$$

is referred to as the **augmentation ideal** of $\mathbb{Z}[G]$.

Definition 1.2.5. The element $N_G = \sum_{\sigma\in G} \sigma$ of $\mathbb{Z}[G]$ is called the **norm** of $\mathbb{Z}[G]$. Furthermore, we define the **coaugmentation** of $\mathbb{Z}[G]$ to be the homomorphism

$$\begin{aligned} \mu : \mathbb{Z} &\to \mathbb{Z}[G] \\ n &\mapsto n \cdot N_G \end{aligned}$$

Its cokernel is denoted

$$J_G = \mathbb{Z}[G]/\mathbb{Z}N_G$$

where $\mathbb{Z}N_G$ is the **coaugmentation ideal** of $\mathbb{Z}[G]$.

Proposition 1.2.6.

1. *I_G is the free abelian group on the set $\{\,\sigma - 1 \mid 1 \neq \sigma \in G\,\}$ and the short exact sequence*

$$0 \longrightarrow I_G \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

splits.

2. *J_G is the free abelian group on the set $\{\,\sigma \pmod{\mathbb{Z}N_G} \mid 1 \neq \sigma \in G\,\}$ and the short exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[G] \longrightarrow J_G \longrightarrow 0$$

splits.

Proof.

Part 1: First observe that, given $x \in I_G$, we have

$$x = \sum_{\sigma \in G} n_\sigma \sigma = \left(\sum_{\sigma \in G} n_\sigma \sigma\right) - \left(\sum_{\sigma \in G} n_\sigma\right) = \sum_{\sigma \in G} n_\sigma(\sigma - 1) = \sum_{1 \neq \sigma \in G} n_\sigma(\sigma - 1)$$

So we get a surjective mapping onto the free abelian group given in the Proposition. To see that this mapping is injective, observe that

$$\begin{aligned} \sum_{1 \neq \sigma \in G} n_\sigma \sigma = 0 \iff & n_\sigma = 0 \text{ for all } 1 \neq \sigma \in G \\ \iff & x = 0 \end{aligned}$$

To see that the exact sequence splits, note that

$$x = \sum_{\sigma \in G} n_\sigma \sigma = \sum_{\sigma \in G} n_\sigma(\sigma - 1) + \sum_{\sigma \in G} n_\sigma$$

which immediately yields an isomorphism $\mathbb{Z}[G] \cong I_G \oplus \mathbb{Z}$.

Part 2: This follows immediately upon dualising the proof for Part 1.

□

Corollary 1.2.7. *We have that $I_G = \operatorname{Ann} \mathbb{Z}N_G$ and $\mathbb{Z}N_G = \operatorname{Ann} I_G$.*

Proof. Fix $\sum_{\sigma \in G} n_\sigma \sigma \in \mathbb{Z}[G]$. We have that

$$\left(\sum_{\sigma \in G} n_\sigma \sigma\right) \cdot N_G = \sum_{\sigma \in G} n_\sigma(\sigma \cdot N_G) = \sum_{\sigma \in G} n_\sigma N_G = \left(\sum_{\sigma \in G} n_\sigma\right) \cdot N_G = 0$$

if and only if $\sum_{\sigma \in G} n_\sigma = 0$ which is exactly what it means for $\sum_{\sigma \in G} n_\sigma \sigma \in \mathbb{Z}[G] \in I_G$.

To prove the second part, we note that by Proposition 1.2.6 we have that

$$\begin{aligned} \sum_{\tau \in G} n_\tau \tau \in \operatorname{Ann}(I_G) \iff & \left(\sum_{\tau \in G} n_\tau \tau\right) \cdot (\sigma - 1) = 0 \quad \text{(for all } 1 \neq \sigma \in G) \\ \iff & \sum_{\tau \in G} n_\tau \tau \sigma = \sum_{\tau \in G} n_\tau \tau \qquad \text{(for all } 1 \neq \sigma \in G) \\ \iff & n_\tau = n_1 \qquad \text{(for all } \tau \in G) \\ \iff & \sum_{\tau \in G} n_\tau \tau = n_1 \cdot N_G \in \mathbb{Z}N_G \end{aligned}$$

□

Definition 1.2.8. Let A be a G-module. Then we define the **norm group** of A to be the G-submodule of A given by

$$N_G A = \{ N_G a \mid a \in A \}$$

Furthermore, we define the following G-submodules of A:

$$\begin{aligned} {}_{N_G}A &= \{ a \in A \mid N_G a = 0 \} \\ I_G A &= \left\{ \sum_{\sigma \in G} n_\sigma (a_\sigma^\sigma - a_\sigma) \,\middle|\, a_\sigma \in A \right\} \end{aligned}$$

We observe that $N_G A \subseteq A^G$ and $I_G A \subseteq {}_{N_G}A$ so we get factor groups $A^G/N_G A$ and ${}_{N_G}A/I_G A$

1.3 Hom-Sets

Definition 1.3.1. Let A and B be G-modules. Then the hom-set $\mathrm{Hom}(A,B) = \mathbf{AbGrp}(A,B)$ consisting of all morphisms of abelian groups between A and B is a G-module with the action defined as follows. Given $\sigma \in G$ and a homomorphism $\phi : A \to B$, define

$$\phi^\sigma = \sigma \circ \phi \circ \sigma^{-1}$$

We write $\mathrm{Hom}_G(A,B) = \boldsymbol{G}_{\mathbf{mod}}(A,B)$ for the subgroup of $\mathrm{Hom}(A,B)$ consisting of all G-homomorphisms between A and B.

Proposition 1.3.2. *Let A and B be G-modules. Then* $\mathrm{Hom}_G(A,B) = \mathrm{Hom}(A,B)^G$.

Proof. We have that

$$\begin{aligned} \phi \in \mathrm{Hom}(A,B)^G &\iff \phi^\sigma = \phi && \text{(for all } \sigma \in G) \\ &\iff \sigma \circ \phi \circ \sigma^{-1} = \phi && \text{(for all } \sigma \in G) \\ &\iff \sigma \circ \phi = \phi \circ \sigma && \text{(for all } \sigma \in G) \\ &\iff \phi \in \mathrm{Hom}_G(A,B) \end{aligned}$$

□

Proposition 1.3.3. *The hom-functor*

$$\mathrm{Hom}_G(-,-) : \boldsymbol{G}_{\mathbf{mod}}^{\mathrm{op}} \times \boldsymbol{G}_{\mathbf{mod}} \to \boldsymbol{G}_{\mathbf{mod}}$$

is additive in both arguments. That is to say, given any family $\{ A_i \}_{i \in I}$ of G-modules and an arbitrary G-module X, we have canonical isomorphisms

$$\begin{aligned} \mathrm{Hom}_G\left(\bigoplus_{i \in I} A_i, X \right) &\cong \prod_{i \in I} \mathrm{Hom}_G(A_i, X) \\ \mathrm{Hom}_G\left(X, \prod_{i \in I} A_i \right) &\cong \prod_{i \in I} \mathrm{Hom}_G(X, A_i) \end{aligned}$$

Moreover, if X can be taken to be finitely generated then

$$\mathrm{Hom}_G\left(X, \bigoplus_{i \in I} A_i \right) \cong \bigoplus_{i \in I} \mathrm{Hom}_G(X, A_i)$$

Proof. This is immediate upon realising that the hom-functor is contravariant in the first argument and covariant in the second; along with the fact that hom-functors preserve all limits in both arguments. In particular, the first argument takes colimits to limits. □

Corollary 1.3.4. *Let X be a G-free module. Then*

$$\mathrm{Hom}_G(X, -) : \boldsymbol{G}_{\mathbf{mod}} \to \boldsymbol{G}_{\mathbf{mod}}$$

is an exact functor.

Proof. Suppose we have an exact sequence

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

in $\boldsymbol{G}_{\mathbf{mod}}$. Write

$$X = \bigoplus_{i \in I} \Gamma_i$$

with each $\Gamma_i \cong \mathbb{Z}[G]$. By Proposition 1.3.3, we have that

$$\mathrm{Hom}_G(X, A) \cong \prod_{i \in I} \mathrm{Hom}_G(\Gamma_i, A)$$

Denote $A_i = \mathrm{Hom}_G(\Gamma_i, A) \cong \mathrm{Hom}_G(\mathbb{Z}[G], A)$. Now, observe that we have an isomorphism

$$\begin{aligned} f : \mathrm{Hom}_G(\mathbb{Z}[G], A) &\to A \\ \phi &\mapsto \phi(1) \end{aligned}$$

The same argumentation yields similar groups B_i and C_i so we get a short exact sequence

$$0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0$$

Since $\boldsymbol{G}_{\mathbf{mod}} = \mathbf{Mod}_{\mathbb{Z}[G]}$ has the property that taking direct sums is an exact functor, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}(X, A) \longrightarrow \mathrm{Hom}(X, B) \longrightarrow \mathrm{Hom}(X, C) \longrightarrow 0$$

as desired. □

Proposition 1.3.5. *Let D be a $\mathbb{Z}$-module. Then any exact sequence*

$$\cdots \longleftarrow X_{q-1} \xleftarrow{d_q} X_q \xleftarrow{d_{q+1}} X_{q+1} \longleftarrow \cdots$$

of $\mathbb{Z}$-free modules induces an exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(X_{q-1}, D) \xrightarrow{d_q^*} \mathrm{Hom}(X_q, D) \xrightarrow{d_{q+1}^*} \mathrm{Hom}(X_{q+1}, D) \longrightarrow \cdots$$

of hom-groups.

Proof. Denote $C_q = \ker d_q = \operatorname{im} d_{q+1}$. Then we have an exact sequence

$$0 \longrightarrow C_q \longrightarrow X_q \longrightarrow C_{q-1} \longrightarrow 0$$

Observe that C_{q-1} is a free subgroup of X_{q-1} so we get a natural homomorphism $\varepsilon : C_{q-1} \to X_q$ satisfying $d_q \circ \varepsilon = \mathrm{id}_{C_{q-1}}$. The Splitting Lemma for $\mathbf{Mod}_{\mathbb{Z}}$ then implies that this exact splits and $X_q = C_q \bigoplus C_{q-1}$.

Now suppose that $f \in \ker d_{q+1}^*$. Then f also vanishes on C_q and so f descends to a homomorphism $g' : C_{q-1} \to D$ on C_{q-1} with $f = g' \circ d_q$. Now, C_{q-1} is a direct summand of X_{q-1} and so g' extends to a homomorphism $g : X_{q-1} \to D$ such that $f = g \circ d_q$. But this is the image of f under the map d_q^* so $f \in \operatorname{im} d_q^*$.

Conversely, suppose that $f \in \operatorname{im} d_q^*$ and let f' be such that $f = d_q^*(f') = f' \circ d_q$. Then $d_{q+1}^*(f) = f \circ d_{q+1} = f \circ d_q \circ d_{q+1} = f \circ 0 = 0$ and so $f \in \ker d_{q+1}^*$. □

1.4 Tensor Products

Definition 1.4.1. Let A and B be G-modules. Then $A \otimes_{\mathbb{Z}} B = A \otimes B$ is also a G-module with action given by

$$(a \otimes b)^\sigma = a^\sigma \otimes b^\sigma$$

for $\sigma \in G$ and a unit tensor $a \otimes b$ and then extending linearly to all of $A \otimes B$.

Proposition 1.4.2. *Let X be a G-module. Then the functor*

$$X \otimes - : \boldsymbol{G}_{\mathbf{mod}} \to \boldsymbol{G}_{\mathbf{mod}}$$

is additive. That is to say, given any family $\{ A_i \}_{i \in I}$ of G-modules then we have a canonical isomorphism

$$X \otimes \left(\bigoplus_{i \in I} A_i \right) = \bigoplus_{i \in I} X \otimes A_i$$

Proof. This is immediate from the fact that taking tensor products commutes with colimits in $\boldsymbol{G}_{\mathbf{mod}} = \mathbf{Mod}_{\mathbb{Z}[G]}$. □

Proposition 1.4.3. *Let A be a free $\mathbb{Z}$-module. Then*

$$- \otimes A : \mathbf{Mod}_{\mathbb{Z}} \to \mathbf{Mod}_{\mathbb{Z}}$$

is an exact functor on exact sequences of free $\mathbb{Z}$-modules.

Proof. Suppose we are given an exact sequence

$$0 \longrightarrow X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \longrightarrow 0$$

of free $\mathbb{Z}$-modules. Then the exactness of the induced sequence

$$X \otimes A \longrightarrow Y \otimes A \longrightarrow Z \otimes A \longrightarrow 0$$

is immediate from the exactness of the original sequence. We just need to show that the induced map

$$\begin{aligned} \phi^* : X \otimes A &\to Y \otimes A \\ x \otimes a &\mapsto \phi(x) \otimes a \end{aligned}$$

is injective. Since Z is free, we can find a natural homomorphism $f : Z \to Y$ such that $\psi \circ f = \mathrm{id}_Z$. The splitting lemma for $\mathbf{Mod}_{\mathbb{Z}}$ then implies that the original exact sequence splits and we have a direct sum decomposition $Y \cong X \oplus Z$. It then follows that

$$Y \otimes A = (X \otimes A) \oplus (Z \otimes A)$$

□

Proposition 1.4.4. *Let X be a free $\mathbb{Z}$-module. Then*

$$X \otimes - : \mathbf{Mod}_{\mathbb{Z}} \to \mathbf{Mod}_{\mathbb{Z}}$$

is an exact functor.

Proof. Fix an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Suppose that $X = \bigoplus_{i \in I} Z_i$ with $Z_i \cong \mathbb{Z}$. By Proposition 1.4.2 we have isomorphisms

$$X \otimes A \cong X \bigoplus_{i \in I} Z_i \otimes A$$

Write $A_i = Z_i \otimes A$. Observe that $Z_i \otimes A \cong \mathbb{Z} \otimes A \cong A$ so that the original exact sequence implies the exactness of

$$0 \longrightarrow X \otimes A_i \longrightarrow X \otimes B_i \longrightarrow X \otimes C_i \longrightarrow 0$$

from which we get an exact sequence

$$0 \longrightarrow X \otimes A \longrightarrow X \otimes B \longrightarrow X \otimes C \longrightarrow 0$$

□

2 Definitions of Tate Cohomology

Throughout this section, G will always be a finite group.

2.1 Completely Free Resolutions

Definition 2.1.1. A **completely free resolution** of G is a commutative diagram

$$\begin{array}{ccccccccccccc}
\cdots & \xleftarrow{d_{-2}} & X_{-2} & \xleftarrow{d_1} & X_{-1} & & \xleftarrow{\quad d_0 \quad} & & X_0 & \xleftarrow{d_1} & X_1 \xleftarrow{d_2} X_2 & \xleftarrow{d_3} & \cdots \\
 & & & & & \nwarrow^{\mu} & & \swarrow_{\varepsilon} & & & & & \\
 & & & & & & \mathbb{Z} & & & & & & \\
 & & & & & \swarrow & & \nwarrow & & & & & \\
 & & & & 0 & & & & 0 & & & &
\end{array}$$

in $\boldsymbol{G}_{\mathbf{mod}}$ which is exact at every term.

Definition 2.1.2. Let $q \geq 1$. We shall refer to the elements of G^q as ***q*-cells** and the individual coordinates of a q-cell as the **vertices** of the cell. We let $X_q = X_{-q-1}$ be the G-free module on all q-cells. In other words

$$X_q = X_{-q-1} = \bigoplus_{\vec{\sigma} \in G^q} \mathbb{Z}[G]\vec{\sigma}$$

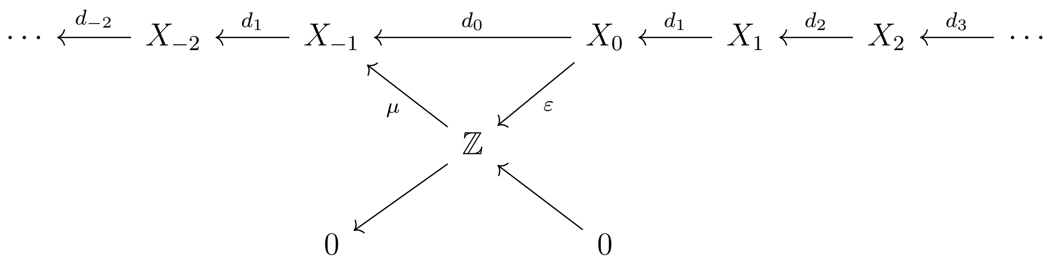
Moreover, we denote

$$X_0 = X_{-1} = \mathbb{Z}[G]$$

and let $\varepsilon : X_0 \to \mathbb{Z}$ and $\mu : \mathbb{Z} \to X_0$ be the augmentation and coaugmentation maps respectively. Finally, we define maps $d_q : X_q \to X_{q-1}$. Since the X_q are all G-free modules, it suffices to define the d_q on the q-cells (and then we may extend linearly):

$$
\begin{aligned}
d_0(1) &= N_G && (q=0)\\
d_1(\sigma) &= \sigma - 1 && (q=1)\\
d_q(\sigma_1,\ldots,\sigma_q) &= \sigma_1(\sigma_2,\ldots,\sigma_q)\\
&\quad + \sum_{i=1}^{q-1}(-1)^i(\sigma_1,\ldots,\sigma_{i-1},\sigma_i\sigma_{i+1},\sigma_{i+2},\ldots,\sigma_q)\\
&\quad + (-1)^q(\sigma_1,\ldots,\sigma_{q-1}) && (q>1)\\
d_{-1}(1) &= \sum_{\sigma\in G}[\sigma^{-1}(\sigma)-\sigma]\\
d_{-q-1}(\sigma_1,\ldots,\sigma_q) &= \sum_{\sigma\in G}\sigma^{-1}(\sigma,\sigma_1,\ldots,\sigma_q)\\
&\quad + \sum_{\sigma\in G}\sum_{i=1}^{q}(-1)^i(\sigma_1,\ldots,\sigma_{i-1},\sigma_i\sigma,\sigma^{-1},\sigma_{i+1},\ldots,\sigma_q)\\
&\quad + \sum_{\sigma\in G}(-1)^{q+1}(\sigma_1,\ldots,\sigma_q,\sigma) && (-q-1<-1)
\end{aligned}
$$

This gives us a diagram



in $\boldsymbol{G}_{\mathbf{mod}}$ which is called the **standard complex** of G.

Proposition 2.1.3. *The standard complex of G is a completely free resolution of G.*

Proof. By construction, each X_q is a free G-module and the ε, μ, d_q are all G-homomorphisms. To see that $\mu\circ\varepsilon = d_0$, observe that

$$(\mu\circ\varepsilon)(1) = \mu(1) = N_G = d_0(1)$$

Since the two functions agree on the generator 1, they must agree everywhere and so $\mu\circ\varepsilon = d_0$. It remains to show that the diagram is exact at each term. We do this by first splitting the complex up into two sequences. The first of which is

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\varepsilon} X_0 \xleftarrow{d_1} X_1 \xleftarrow{d_2} X_2 \xleftarrow{d_3} \cdots \tag{1}$$

Let $i : \mathbb{Z}\to X_0$ denote the inclusion and define the maps

$$
\begin{aligned}
D_0 : X_0 &\to X_1\\
\sigma &\mapsto (\sigma)
\end{aligned}
$$

$$
\begin{aligned}
D_q : X_q &\to X_{q+1}\\
\sigma(\sigma_1,\ldots,\sigma_q) &\mapsto (\sigma,\sigma_1,\ldots,\sigma_q)
\end{aligned}
$$

After some elementary calculations, we get

$$i \circ \varepsilon + d_1 \circ D_0 = \mathrm{id}_{X_0}$$
$$D_{q-1} \circ d_q + d_{q+1} \circ D_q = \mathrm{id}_{X_q}$$

Now if $x \in \ker \varepsilon$, the first equation implies that $x \in \operatorname{im} d_1$. Conversely, suppose that $x \in \operatorname{im} d_1$. Now, it is immediate that $\varepsilon \circ d_1 = 0$ and so $\operatorname{im} d_1 \subseteq \ker d_1$ whence the sequence is exact at $\mathbb{Z}$.

Similarly, if $x \in \ker d_q$ then $x \in \operatorname{im} d_{q+1}$. To prove the inclusion in the opposite direction, we shall prove that $d_q \circ d_{q+1} = 0$ by induction on $q \geq 1$. When $q = 0$ we set $d_0 = \varepsilon$ and D_{-1} by i. Then the basis case is clear. Assume that we have $d_{q-1} \circ d_q = 0$. On one hand we have

$$d_q = (D_{q-2} \circ d_{q-1} + d_q \circ D_{q-1}) \circ d_q = d_q \circ D_{q-1} \circ d_q$$

On the other we have

$$d_q = d_q \circ (D_{q-1} \circ d_q + d_{q+1} \circ D_q) = d_q \circ D_{q-1} \circ d_q + d_q \circ d_{q+1} \circ D_q$$

Subtracting these equations gives

$$d_q \circ d_{q+1} \circ D_q = 0$$

But every cell in X_{q+1} is in the image of D_q so we conclude that $d_q \circ d_{q+1} = 0$. This completes the proof of exactness at X_q.

The second sequence is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\mu} X_{-1} \xrightarrow{d_{-1}} X_{-2} \xrightarrow{d_{-2}} X_{-3} \xrightarrow{d_{-3}} \cdots$$

the exactness of which follows by dualising the above argument in the following way. Taking $\mathrm{Hom}(-, \mathbb{Z})$ of Sequence 1 yields a sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(X_0, \mathbb{Z}) \longrightarrow \mathrm{Hom}(X_1, \mathbb{Z}) \longrightarrow \mathrm{Hom}(X_2, \mathbb{Z}) \longrightarrow \cdots$$

which is exact by Proposition 1.3.5. Now if $\mathcal{X}_q = \{ x_i \}$ is the system of generators of X_q consisting of all q-cells, let $\mathcal{X}^{q*}$ be the so-called dual system of generators consisting of the dual basis elements

$$x_i^*(\sigma x_k) = \begin{cases} 1 & \text{if } \sigma = 1, i = k \\ 0 & \text{if otherwise} \end{cases}$$

is a $\mathbb{Z}[G]$-free generators of $\mathrm{Hom}(X_q, \mathbb{Z})$. If we identify each x_i with x_i^* then we get a canonical G-isomorphism $X_{-q-1} \cong \mathrm{Hom}(X_q, \mathbb{Z})$ which shows that the second sequence is indeed exact.

The last thing we need to check is exactness of the sequence

$$X_{-2} \xleftarrow{d_{-1}} X_{-1} \xleftarrow{d_0} X_0 \xleftarrow{d_1} X_1$$

Observe that μ is injective, ε is surjective and $d_0 = \mu \circ \varepsilon$. So $\ker d_0 = \ker \varepsilon$ and $\operatorname{im} d_0 = \operatorname{im} \mu$ whence $\ker d_0 = \operatorname{im} d_1$ and $\ker d_{-1} = \operatorname{im} d_0$. □

Definition 2.1.4. Let A be a G-module. We define the ***q*-cochains** of A to be the group

$$A_q = \mathrm{Hom}_G(X_q, A)$$

We also have a natural map

$$\partial_q : \mathrm{Hom}_G(X_{q-1}, A) \to \mathrm{Hom}_G(X_q, A)$$
$$\phi \mapsto \phi \circ d_q$$

Proposition 2.1.5. *Let A be a G-module. Then the sequence*

$$\cdots \xrightarrow{\partial_{-2}} A_{-2} \xrightarrow{\partial_{-1}} A_{-1} \xrightarrow{\partial_0} A_0 \xrightarrow{\partial_1} A_1 \xrightarrow{\partial_2} A_2 \xrightarrow{\partial_3} \cdots$$

is a cochain complex in **AbGrp**.

Proof. By definition, we need to show that for every q we have $\partial_{q+1} \circ \partial_q = 0$. To this end, fix $q \geq 0$ and $\phi \in A_q$. Then

$$(\partial_{q+1} \circ \partial)(\phi) = \partial_{q+1}(\phi \circ d_q) = \phi \circ d_q \circ d_{q+1}$$

But $d_q \circ d_{q+1} = 0$ since they are part of a complete free resolution of G. □

Remark. Since the q-cochains are uniquely determined by their values on q-cells, we can identify A_q with the collection of all maps $G^q \to A$.

Definition 2.1.6. Let A be a G-module. We define the ***q*-cocycles** to be $Z_q = \ker \partial_{q+1}$ and the ***q*-coboundaries** to be $R_q = \operatorname{im} \partial_q$. We then define the **Tate cohomology group of dimension *q*** to be

$$H^q(G, A) = {Z_q}/{R_q}$$

We shall also refer to $H^q(G, A)$ as the ***q*-th cohomology group with coefficients in *A***.

2.2 Explicit Descriptions of Low Dimensional Objects

2.2.1 $H^{-1}(G, A)$

We have the following explicit descriptions for the -1-dimensional objects:

$$\begin{aligned} A_{-1} &= \operatorname{Hom}_G(X_{-1}, A) = \operatorname{Hom}_G(\mathbb{Z}[G], A) = A \\ Z_{-1} &= \ker \partial_0 = {}_{N_G}A \\ R_{-1} &= I_G A \\ H^{-1}(G, A) &= {}_{N_G}A / I_G A \end{aligned}$$

2.2.2 $H^0(G, A)$

We have the following explicit descriptions for the 0-dimensional objects:

$$\begin{aligned} A_0 &= \operatorname{Hom}_G(X_0, A) = \operatorname{Hom}_G(\mathbb{Z}[G], A) = A \\ Z_0 &= \ker \partial_1 = A^G \\ R_0 &= N_G A \\ H^0(G, A) &= A^G / N_G A \end{aligned}$$

We refer to $H^0(G, A)$ as the **norm residue group** of the G-module A.

2.2.3 $H^1(G, A)$

The 1-cochains form the group $\mathrm{Hom}_G(A_1, A)$ which coincides with all functions $f : G \to A$.

The 1-cocyles are all 1-cochains $x : G \to A$ satisfying $\partial\partial_2 x = 0$. In other words, they are the 1-cochains that satisfy the **crossed homomorphism** condition

$$x(\sigma\tau) = x(\tau)^\sigma + x(\sigma)$$

for all $\sigma, \tau \in G$.

The 1-coboundaries are all the 1-cochains $x : G \to A$ such that there exists some 0-cochain $a \in A$ with $\partial_1 x = a$. In other words

$$x(\sigma) = a^\sigma - a$$

for some $a \in A$.

Observe that if the G-action on A is trivial then the crossed homomorphisms are exactly the homomorphisms $G \to A$. Moreover, there are no non-trivial 1-coboundaries. Hence, in this case, $H^1(G, A) = \mathrm{Hom}(G, A)$.

Adding to the previous remark, consider $\mathbb{Q}/\mathbb{Z}$ as G-module with the trivial action of G. Then $H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) = \widehat{G}$ is the character group of G.

2.2.4 $H^2(G, A)$

The 2-cochains form the group $\mathrm{Hom}_G(A_2, A)$ which coincides with all functions $f : G^2 \to A$.

The 2-cocycles are all the 2-cochains $x : G^2 \to A$ satisfying $\partial_3 x = 0$. In other words, they are the 2-cochains satisfying the **factor system** condition

$$x(\sigma\tau, \rho) + x(\sigma, \tau) = x(\tau, \rho)^\sigma + x(\sigma, \tau\rho)$$

for all $\sigma, \tau, \rho \in G$.

The 2-coboundaries are all the 2-cochains $x : G^2 \to A$ such that

$$x(\sigma, \tau) = y(\tau)^\sigma - y(\sigma\tau) + y(\sigma)$$

for some 1-cochain $y : G \to A$.

Factor systems are related to the problem of group extensions.

Definition 2.2.1. Let G be a group. We say that $\widehat{G}$ is a **group extension** of G if $\widehat{G}$ has a subgroup isomorphic to G.

Now suppose that we are given a multitplicative abelian group A and an arbitrary group G. We want to find all group extensions $\widehat{G}$ of A such that A is normal in $\widehat{G}$ and $\widehat{G}/A \cong G$.

Assume that we have a solution $\widehat{G}$ to the posed problem. Let $\{ u_\sigma \}$ be a complete set of coset representatives of $\widehat{G}/A \cong G$ so that each element of $\widehat{G}$ can be written as $a \cdot u_\sigma$ for some $a \in A$ and $\sigma \in G$. In order to determine the group table of $\widehat{G}$, we need to be able to express $u_\sigma \cdot a$ and $u_\sigma \cdot u_\tau$ for some $\sigma, \tau \in G$ in the aforementioned form.

Since A is normal in $\widehat{G}$, $u_\sigma \cdot a$ is in the same right coset as u_σ. Hence there exists $a^\sigma \in A$ such that $u_\sigma \cdot a = a^\sigma \cdot u_\sigma$. This defines the structure of a a G-module on A via the assignment $a \mapsto a^\sigma = u_\sigma \cdot a \cdot u_\sigma^{-1}$.

Now fix $\sigma, \tau \in G$. Then the product $u_\sigma \cdot u_\tau$ lies in the same right coset as $u_{\sigma\tau}$. In other words, $u_\sigma \cdot u_\tau = x(\sigma, \tau) \cdot u_{\sigma\tau}$ for some $x(\sigma, \tau) \in A$. Now observe that

$$(u_\sigma \cdot u_\tau) \cdot u_\rho = x(\sigma, \tau) \cdot u_{\sigma\tau} \cdot u_\rho = x(\sigma, \tau) \cdot x(\sigma\tau, \rho) \cdot u_{\sigma\tau\rho}$$

and on the other hand

$$u_\sigma \cdot (u_\tau \cdot u_\rho) = u_\sigma \cdot x(\tau, \rho) \cdot u_{\tau\rho} = x(\tau, \rho)^\sigma \cdot u_\sigma \cdot u_{\tau\rho} = x(\tau, \rho) \cdot x(\sigma, \tau\rho) \cdot u_{\sigma\tau\rho}$$

Comparing these two, we then have that

$$x(\sigma, \tau) \cdot x(\sigma\tau, \rho) = x(\tau, \rho) \cdot x(\sigma, \tau\rho)$$

which is exactly the factor system condition and so x is a 2-cocycle.

Now suppose that $\{ u'_\sigma \}$ is another set of coset representatives of $\widehat{G}/A = G$. Then, from the above analysis, we get another factor system $x'(\sigma, \tau)$. Observe that, given $\sigma \in G$ we have that $u'_\sigma \cdot u_\sigma^{-1} \in A$. Moreover,

$$\begin{aligned} u'_{(-)} \cdot u^{-1}_{(-)} : G^2 &\to A \\ (\sigma, \tau) &\mapsto u'_\sigma \cdot u_\sigma^{-1} \end{aligned}$$

is a 2-cocycle. Since A is abelian, we then have that

$$\begin{aligned} \frac{u'_\sigma u'_\tau}{u_\sigma u_\tau} &= \frac{x'(\sigma, \tau) u'_{\sigma\tau}}{x(\sigma, \tau) u_{\sigma\tau}} \\ u'_\sigma u'_\tau u_\tau^{-1} u_\sigma^{-1} &= x'(\sigma, \tau) u'_{\sigma\tau} u^{-1}_{\sigma\tau} x(\sigma, \tau)^{-1} \\ u'_\sigma u'_\tau u_\tau^{-1} u_\sigma^{-1} &= u'_{\sigma\tau} u^{-1}_{\sigma\tau} \frac{x'(\sigma, \tau)}{x(\sigma, \tau)} \\ u'_\sigma u'_\tau u_\tau^{-1} u_\sigma^{-1} u'_{\sigma\tau} u^{-1}_{\sigma\tau} &= \frac{x'(\sigma, \tau)}{x(\sigma, \tau)} \\ u'_\sigma u'_\tau u_\tau^{-1} ({u'_\sigma}^{-1} u'_\sigma) u_\sigma^{-1} u'_{\sigma\tau} u^{-1}_{\sigma\tau} &= \\ u'_\sigma u'_\tau u_\tau^{-1} {u'_\sigma}^{-1} u'_{\sigma\tau} u^{-1}_{\sigma\tau} (u'_\sigma u_\sigma^{-1}) &= \\ (u'_\tau u_\tau^{-1})^\sigma \cdot (u_{\sigma\tau' u_{\sigma\tau}})^{-1} \cdot (u'_\sigma u_\sigma^{-1}) &= \end{aligned}$$

which is exactly the 2-coboundary condition in multiplicative notation. This shows that $\widehat{G}$ is uniquely determined by the conjugation action of G on A and a class of equivalent factor systems $x(\sigma, \tau)$ up to 2-coboundaries: a cohomology class in $H^2(G, A)$.

Conversely, suppose that A is a G-module and that we have a cohomology class $c \in H^2(G, A)$. Then this information determines a group extension $\widehat{G}$ of A in the following way. $\widehat{G}$ is the free group with generators u_σ for $\sigma \in G$ and the elements of A subject to the relations

$$a^\sigma = u_\sigma \cdot a u_\sigma^{-1}, \qquad u_\sigma \cdot u_\tau = x(\sigma, \tau) \cdot u_{\sigma\tau}$$

where $x(\sigma, \tau)$ is an element of c.

3 Properties of Cohomology Groups

Throughout this section, G will always be a finite group.

3.1 Basic Properties

Proposition 3.1.1. *Let $f : A \to B$ be a morphism of G-modules. Then f induces a canonical homomorphism*

$$\overline{f_q} : H^q(G, A) \to H^q(G, B)$$

given by post-composition with f.

Proof. We first define the map

$$\begin{aligned} f_q : A_q &\to B_q \\ \phi &\mapsto f \circ \phi \end{aligned}$$

Then it is clear that f_q is a homomorphism between q-cochains. Since f is a G-homomorphism, it commutes with the action of G and, in particular, we have that $\partial_{q+1} \circ f_q = f_{q+1} \circ \partial_{q+1}$. Now suppose that ϕ is a q-cocycle with respect to A. Then

$$\begin{aligned} \phi \in \ker \partial_{q+1} \iff \partial_{q+1}(\phi) = 0 \implies (f_{q+1} \circ \partial_{q+1})(\phi) = 0 &\iff (\partial_{q+1} \circ f_q)(\phi) = 0 \\ &\iff f_q(\phi) \in \ker \partial_{q+1} \end{aligned}$$

so that $f_q(\phi)$ is a q-cocycle with respect to B. Now assume that ϕ is a q-coboundary with respect to A. Then

$$\begin{aligned} \phi \in \operatorname{im} \partial_q \iff \partial_q(\psi) = \phi \text{ for some } \psi \in A_{q-1} &\implies (f_q \circ \partial_q)(\psi) = f_q(\phi) \\ &\iff (\partial_q \circ f_{q-1})(\psi) = f_q(\phi) \\ &\iff f_q(\phi) \in \operatorname{im} \delta_q \end{aligned}$$

so that $f_q(\phi)$ is a q-coboundary with respect to B. It then follows that f_q induces a homomorphism of cohomology groups

$$\begin{aligned} &\overline{f_q} : H^q(G, A) \to H^q(G, B) \\ &\phi \pmod{R_q} \mapsto f_q(\phi) \pmod{f_q(R_q)} \end{aligned}$$

□

Proposition 3.1.2. *Let*

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

be a short exact sequence of G-modules. Then there exists a canonical homomorphism

$$\delta_q : H^q(G, C) \to H^{q+1}(G, A)$$

of cohomology groups called the ***connecting homomorphism****.*

Proof. Conisder the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{q-1} & \xrightarrow{\phi_{q-1}} & B_{q-1} & \xrightarrow{\psi_{q-1}} & C_{q-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial_q} & & \downarrow{\scriptstyle \partial_q} & & \downarrow{\scriptstyle \partial_q} & & \\ 0 & \longrightarrow & A_q & \xrightarrow{\phi_q} & B_q & \xrightarrow{\psi_q} & C_q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial_{q+1}} & & \downarrow{\scriptstyle \partial_{q+1}} & & \downarrow{\scriptstyle \partial_{q+1}} & & \\ 0 & \longrightarrow & A_{q+1} & \xrightarrow{\phi_{q+1}} & B_{q+1} & \xrightarrow{\psi_{q+1}} & C_{q+1} & \longrightarrow & 0 \end{array}$$

which is obtained by applying the functor $\mathrm{Hom}_G(X_i, -)$ to the exact sequence. Since the X_i are free G-modules, Corollary 1.3.4 then implies that the rows of this diagram are exact.

We shall write a_q, b_q, c_q for the q-cochains in A_q, B_q and C_q respectively. We shall write $\overline{a_q}, \overline{b_q}, \overline{c_q}$ for their corresponding images in the cohomology groups.

Suppose that we are given $\overline{c_q} \in H^q(G, C)$. We want to define $\delta_q(\overline{c_q})$. Since c_q is a q-cochain with respect to C, we know that $\partial_{q+1}(c_q) = 0$. Moreover, the map ψ_q is surjective so we can always choose a b_q such that $\psi_q(b_q) = c_q$. Then

$$(\psi_{q+1} \circ \partial_{q+1})(b_q) = (\partial_{q+1} \circ \psi_q)(b_q) = \partial_{q+1}(c_q) = 0$$

and so $\partial_{q+1}(b_q) \in \ker \psi_{q+1} = \mathrm{im}\, \phi_{q+1}$. Hence there exists a_{q+1} such that $\partial_{q+1}(b_q) = \phi_{q+1}(a_{q+1})$. Since the X_i form a completeley free resolution of G, we have that $\partial_{q+1} \circ \partial_q = 0$ and so

$$(\phi_{q+2} \circ \partial_{q+2})(a_{q+1}) = (\partial_{q+2} \circ \phi_{q+1})(a_{q+1}) = (\partial_{q+2} \circ \partial_{q+1})(b_q) = 0$$

But ϕ_{q+2} is injective and so $\partial_{q+2}(a_{q+1}) = 0$ whence a_{q+1} is a $(q+1)$-cochain with respect to A. We then define

$$\begin{aligned} \delta_q : H^q(G, C) &\to H^{q+1}(G, C) \\ \overline{c_q} &\mapsto \overline{a_{q+1}} \end{aligned}$$

It remains to show that this definition of δ_q is well-defined. In other words, we must show that it is independent of the choice of representative c_q of $\overline{c_q}$ and preimage b_q. To this end, suppose that c'_q is another representative and b'_q is another preimage. Let $\overline{a'_{q+1}}$ denote the corresponding $(q+1)$-cochain. Then

$$\begin{aligned} \overline{c_q} = \overline{c'_q} &\implies c_q - c'_q = \partial_q(c_{q-1}) && \text{(for some } c_{q-1}\text{)} \\ &\implies c_q - c'_q = (\partial_q \circ \psi_{q-1})(b_{q-1}) && \text{(for some } b_{q-1}\text{)} \\ &\implies \psi_q(b_q) - \psi_q(b'_q) = (\psi_q \circ \partial_q)(b_{q-1}) \\ &\implies b_q - b'_q - \partial_q(b_{q-1}) \in \ker \psi_q = \mathrm{im}\, \phi_q \\ &\implies \phi_q(a_q) = b_q - b'_q - \partial_q(b_{q-1}) && \text{(for some } a_q\text{)} \\ &\implies (\partial_{q+1} \circ \phi_q)(a_q) = \partial_q(b_q) - \partial_q(b'_q) \\ &\implies (\phi_{q+1} \circ \partial_{q+1})(a_q) = \phi_{q+1}(a_{q+1}) - \phi_{q+1}(a'_{q+1}) \\ &\implies \partial_{q+1}(a_q) = a_{q+1} - a'_{q+1} && (\phi_{q+1} \text{ is injective}) \\ &\implies \overline{a_{q+1}} = \overline{a'_{q+1}} \end{aligned}$$

□

Theorem 3.1.3. *Let*

$$0 \longrightarrow A \xrightarrow{\ \phi\ } B \xrightarrow{\ \psi\ } C \longrightarrow 0$$

be a short exact sequence of G-modules. Then there exists a long exact sequence of cohomology groups.

$$\begin{array}{ccccc}
H^{-q}(G,A) & \longrightarrow & H^{-q}(G,B) & \longrightarrow & H^{-q}(G,C) \\
& & \vdots & & \\
H^{-1}(G,A) & \longrightarrow & H^{-1}(G,B) & \longrightarrow & H^{-1}(G,C) \\
H^{0}(G,A) & \longrightarrow & H^{0}(G,B) & \longrightarrow & H^{0}(G,C) \\
H^{1}(G,A) & \longrightarrow & H^{1}(G,B) & \longrightarrow & H^{1}(G,C) \\
& & \vdots & & \\
H^{q}(G,A) & \longrightarrow & H^{q}(G,B) & \longrightarrow & H^{q}(G,C)
\end{array}$$

Proof. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_q & \xrightarrow{\phi_q} & B_q & \xrightarrow{\psi_q} & C_q & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \partial_{q+1}} & & \downarrow{\scriptstyle \partial_{q+1}} & & \downarrow{\scriptstyle \partial_{q+1}} & & \\
0 & \longrightarrow & A_{q+1} & \xrightarrow{\phi_{q+1}} & B_{q+1} & \xrightarrow{\psi_{q+1}} & C_{q+1} & \longrightarrow & 0
\end{array}$$

Let Z_q^A denote the q-cochains with respect to A and similarly for B and C. Let Q_q^A represent the cokernel of the map ∂_q with respect to A and similarly for B and C. By the Snake Lemma, we then have an exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & Z_q^A & \longrightarrow & Z_q^B & \longrightarrow & Z_q^C \\
Q_{q+1}^A & \longrightarrow & Q_{q+1}^B & \longrightarrow & Q_{q+1}^C & \longrightarrow & 0
\end{array}$$

Shifting the dimensions as necessary, we get a commutative diagram

$$\begin{array}{ccccccc}
& & Q_q^A & \longrightarrow & Q_q^B & \longrightarrow & Q_q^C & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle m_q^A} & & \downarrow{\scriptstyle m_q^B} & & \downarrow{\scriptstyle m_q^C} & & \\
0 & \longrightarrow & Z_{q+1}^A & \longrightarrow & Z_{q+1}^B & \longrightarrow & Z_{q+1}^C & &
\end{array}$$

where

$$\begin{aligned}
m_q^A : Q_q &\to Z_{q+1} \\
[a_q] &\mapsto \partial_{q+1}(a_q)
\end{aligned}$$

and similarly for B and C. Clearly, $\ker m_q^A = H^q(G,A)$ and $\operatorname{coker} m_q^A = H^{q+1}(G,A)$ with the same equalities holding for B and C. Appealing to the Snake Lemma once more yields the long exact sequence

$$\begin{array}{ccccc}
H^{q}(G,A) & \longrightarrow & H^{q}(G,B) & \longrightarrow & H^{q}(G,C) \\
H^{q+1}(G,A) & \longrightarrow & H^{q+1}(G,B) & \longrightarrow & H^{q+1}(G,C)
\end{array}$$

as desired. □

Proposition 3.1.4. *Let*

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

be a short exact sequence of G-modules. Then there exists a long exact sequence

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \longrightarrow H^1(G,A) \longrightarrow H^1(G,B) \longrightarrow H^1(G,C)$$

Proof. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_A} & & \downarrow{\scriptstyle d_B} & & \downarrow{\scriptstyle d_C} & & \\ 0 & \longrightarrow & Z_1^A & \xrightarrow{\phi_1} & Z_1^B & \xrightarrow{\psi_1} & Z_1^C & & \end{array}$$

where the exactness of the second row is ensured by a similar argument to the previous proof and

$$\begin{aligned} d_A : A &\to Z_1 \\ a &\mapsto (\sigma \mapsto a^\sigma - a) \end{aligned}$$

Then $\ker d_A = A^G$, $\operatorname{coker} d_A = H^1(G,A)$ and similarly for B and C. Appealing to the Snake Lemma then yields the desired long exact sequence. □

Proposition 3.1.5. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \xrightarrow{\phi'} & B' & \xrightarrow{\psi'} & C' & \longrightarrow & 0 \end{array}$$

be a commutative diagram in $\boldsymbol{G}_{\mathbf{mod}}$ *with exact rows. Then the diagram*

$$\begin{array}{ccc} H^q(G,C) & \xrightarrow{\delta_q} & H^{q+1}(G,A) \\ \downarrow{\scriptstyle \overline{h_q}} & & \downarrow{\scriptstyle \overline{f_{q+1}}} \\ H^q(G,C') & \xrightarrow{\delta_q} & H^{q+1}(G,A') \end{array}$$

commutes.

Proof. Fix $\overline{c_q} \in H^q(G,C)$. Let b_q and a_{q+1} be such that $c_q = \psi(b_q)$ and $\phi(a_{q+1}) = \partial_{q+1}(b_q)$. Then $\delta_q(\overline{c_q}) = \overline{a_{q+1}}$ so that

$$(\overline{f_{q+1}} \circ \delta_q)(\overline{c_q}) = \overline{f_{q+1}}(\overline{a_{q+1}})$$

Let $c'_q = h_q(c_q), b'_q = g_q(b_q)$ and $a'_{q+1} = f_{q+1}(a_{q+1})$. Then $c'_q = \psi'(b'_q)$ and $\partial_{q+1}(b'_q) = \phi'(a'_{q+1})$ so that

$$(\delta_q \circ \overline{h_q})(\overline{c_q}) = \delta_q(\overline{c'_q}) = \overline{a'_{q+1}} = \overline{f_q}(\overline{a'_{q+1}}) = (\overline{f_q} \circ \delta_q)(\overline{c_q})$$

and so the diagram commutes. □

Proposition 3.1.6. *Let*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B' & \longrightarrow & B'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

be a commutative diagram $\boldsymbol{G}_{\mathbf{mod}}$ *with exact rows and columns. Then the diagram*

$$
\begin{array}{ccc}
H^{q-1}(G, C'') & \xrightarrow{\delta_{q-1}} & H^q(G, C') \\
\Big\downarrow{\scriptstyle \delta_{q-1}} & & \Big\downarrow{\scriptstyle -\delta_q} \\
H^q(G, A'') & \xrightarrow{\delta_q} & H^{q+1}(G, A')
\end{array}
$$

commutes.

Proof. Let D be the kernel of the map $B \to C''$ so that we have an exact sequence

$$0 \longrightarrow D \longrightarrow B \longrightarrow C'' \longrightarrow 0$$

Define G-homomorphisims

$$
\begin{aligned}
i : A' &\to A \oplus B' \\
a' &\mapsto (a, b')
\end{aligned}
$$

where a is the image of a' in A and b' is the image of a' in B' and

$$
\begin{aligned}
j : A \oplus B' &\to D \\
(a, b') &\mapsto d_1 - d_2
\end{aligned}
$$

where d_1 is the image of a in D and similarly for b' and d_2. Then we have an exact sequence

$$0 \longrightarrow A' \xrightarrow{i} A \oplus B' \xrightarrow{j} D \longrightarrow 0$$

and a commutative diagram

$$
\begin{array}{ccccccccc}
A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C'' \\
{\scriptstyle \mathrm{id}}\Big\uparrow & & {\scriptstyle (\mathrm{id},0)}\Big\uparrow & & \vdots\uparrow & & \Big\uparrow & & {\scriptstyle \mathrm{id}}\Big\uparrow \\
A' & \xrightarrow{i} & A \oplus B' & \xrightarrow{j} & D & \longrightarrow & B & \longrightarrow & C'' \\
{\scriptstyle -\mathrm{id}}\Big\downarrow & & {\scriptstyle (-\mathrm{id},0)}\Big\downarrow & & \vdots\downarrow & & \Big\downarrow & & {\scriptstyle \mathrm{id}}\Big\downarrow \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C''
\end{array}
$$

By exactness, $\mathrm{im}(D \to B'') \subseteq \mathrm{im}(A'' \to B'')$. Moreover, the map $A'' \to B''$ is injective by hypothesis so we can extend the diagram by a homomorphism $D \to A''$. A similar argument shows that we can extend the diagram by a homomorphism $D \to C'$. This extended diagram is still commutative so applying Proposition 3.1.5 yields a commutative diagram of cohomology groups

$$\begin{array}{ccccc}
H^{q-1}(G,C'') & \xrightarrow{\delta_{q-1}} & H^q(G,A'') & \xrightarrow{\delta_q} & H^{q+1}(G,A') \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\
H^{q-1}(G,C'') & \xrightarrow{\delta_{q-1}} & H^q(G,D) & \xrightarrow{\delta_q} & H^{q+1}(G,A') \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle -\mathrm{id}} \\
H^{q-1}(G,C'') & \xrightarrow{\delta_{q-1}} & H^q(G,C') & \xrightarrow{\delta_q} & H^{q+1}(G,A')
\end{array}$$

The red arrows then yield the desired diagram in the statement of the Proposition. □

Proposition 3.1.7. *The cohomology functor*

$$H^q(G,-) : \mathbf{AbGrp} \to \mathbf{AbGrp}$$

is (co)additive. That is to say, given any family $\{ A_i \}_{i\in I}$ *of* G*-modules, we have canonical isomorphisms*

$$H^q\left(G, \bigoplus_{i\in I} A_i\right) \cong \bigoplus_{i\in I} H^q(G,A_i)$$
$$H^q\left(G, \prod_{i\in I} A_i\right) \cong \prod_{i\in I} H^q(G,A_i)$$

Proof. Let $A = \bigoplus_{i\in I} A_i$. By Proposition 1.3.3 we have

$$A_q = \mathrm{Hom}_G(X_q, A) \cong \bigoplus_{i\in A} \mathrm{Hom}_G(X_q, A_i) = \bigoplus_{i\in A} (A_i)_q$$

and so $Z_q^A = \bigoplus_{Z_q^{A_i}}$ and $R_q^A = \bigoplus_{i\in I} R_q^{A_i}$ whence the cohomology groups also coincide. A similar proof shows that the functor also commutes with products. □

3.2 G-induced Modules

Definition 3.2.1. Let A be a G-module. We say that A is ***G*-induced** if

$$A = \bigoplus_{\sigma\in G} D^\sigma$$

for some subgroup $D \subseteq A$.

Proposition 3.2.2. *Let* A *be a* G*-induced module so that* $A = \bigoplus_{\sigma\in G} D^\sigma$ *for some subgroup* $D \subseteq A$*. Then* $A \cong \mathbb{Z}[G] \otimes D$*.*

Proof. We have that

$$\mathbb{Z}[G] \otimes D = \left(\bigoplus_{\sigma\in G} \mathbb{Z}\sigma\right) \otimes D = \bigoplus_{\sigma\in G} (\mathbb{Z}\otimes D)^\sigma \cong \bigoplus_{\sigma\in G} D^\sigma = A$$

□

Proposition 3.2.3. *Let* X *be a* G*-induced module and* A *a* G*-module. Then* $X \otimes A$ *is a* G*-induced module.*

Proof. Let $D \subseteq X$ be a subgroup such that $X = \bigoplus_{\sigma\in G} \sigma D$. Then

$$X \otimes A = \left(\bigoplus_{\sigma\in G} D^\sigma\right) \otimes A \cong \bigoplus_{\sigma\in G} D^\sigma \otimes \bigoplus_{\sigma\in G} A^\sigma = \bigoplus_{\sigma\in G} (D \otimes A)^\sigma$$

since $D \otimes A$ is a subgroup of $X \otimes A$, this completes the proof. □

Proposition 3.2.4. *Let A be a G-induced module and $H \subseteq G$ a subgroup. Then A is an H-induced H-module. Moreover, if H is normal in G then A^H is a G/H-induced G/H-module.*

Proof. Write $A = \bigoplus_{\sigma\in G} D^\sigma$ for some subgroup $D \subseteq G$. Let $\{\tau_i\}$ be a set of right coset representatives of H in G. Then

$$A = \bigoplus_{\sigma\in H} \bigoplus_{\tau_i} D^{\sigma\tau_i} = \bigoplus_{\sigma\in H} \left(\bigoplus_{\tau_i} D_i^\tau\right)^\sigma$$

so that A is an H-induced H module.

Now suppose that H is normal in G. We claim that the G/H-module A^H satisfies

$$A^H = \sum_{\tau\in G/H} (N_H D)^\tau$$

The sum on the right hand side is clearly a direct sum since A can be expressed as one. Furthermore, any element of $(N_H D)^\tau$ is an element of A^H so the sum is a subset of A^H. Conversely, fix $a \in A^H$. Since a is G-induced, a admits a unique decomposition

$$a = \sum_{\tau\in G} d_\tau^\tau$$

for some $d_\tau \in H$. Now, given $\sigma \in H$, we have

$$a = a^\sigma = \sum_{\tau\in G} d_\tau^{\sigma\tau} = \sum_{\tau\in G} d_{\sigma\tau}^{\sigma\tau} = \sum_{\tau\in G} d_{\sigma\tau}^{\tau} = a$$

By uniqueness, we then have that $d_{\sigma\tau} = d_\tau$. It then follows that

$$a = \sum_{\tau_i} \sum_{\sigma\in H} d_{\tau\sigma}^{\tau\sigma} = \sum_{\tau_i} \left(\sum_{\sigma\in H} d_\tau^\sigma\right)^\tau = \sum_{\tau_i} (N_H d_\tau)^\tau$$

where τ_i ranges over a set of right coset representatives of G/H. This proves the other inclusion. Hence A^H is G/H-induced. □

Definition 3.2.5. Let A be a G-module. We say that A has **trivial cohomology** if

$$H^q(H, A) = 0$$

for all subgroups $H \subseteq G$.

Theorem 3.2.6. *Let A be a G-induced module. Then A has trivial cohomology.*

Proof. By Proposition 3.2.4 it suffices to show that $H^q(G, A) = 0$. In other words, we need to show that the sequence

$$\cdots \longrightarrow \mathrm{Hom}_G(X^q, A) \xrightarrow{\partial_q} \mathrm{Hom}_G(X^{q+1}, A) \longrightarrow \cdots$$

is exact. Suppose that A admits the decomposition $A = \bigoplus_{\sigma \in G} D^\sigma$. Let $\pi : A \to D$ be the projection map given by projecting A onto the coordinate corresponding to the identity of G. Then π induces an isomorphism

$$\begin{aligned} \pi^* : \mathrm{Hom}_G(X_q, A) &\to \mathrm{Hom}(X_q, D) \\ f &\mapsto \pi \circ f \end{aligned}$$

Indeed, this is clearly a homomorphism. To see that it is surjective, given $f \in \mathrm{Hom}(X_q, D)$, let $f^* : X_q \to A$ be the unqiue function limnearly extending f. Then f^* commutes with the action of G and satisfies $\pi^*(f^*) = f$ by construction and. To see that it is injective, note that the image of a function in $\mathrm{Hom}_G(X_q, A)$ is determined uniquely by the image of $\pi \circ f$ so if $\pi \circ f = 0$ we must have that $f = 0$.

Now, Proposition 1.3.5 implies that the sequence

$$\cdots \longrightarrow \mathrm{Hom}(X^q, D) \longrightarrow \mathrm{Hom}(X^{q+1}, D) \longrightarrow \cdots$$

is exact. This, together with the isomorphism π^*, implies that the first sequence is exact as claimed. ☐

Lemma 3.2.7. *Let A be a G-module. Then we have exact sequences*

$$0 \longrightarrow I_G \otimes A \longrightarrow \mathbb{Z}[G] \otimes A \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow A \longrightarrow \mathbb{Z}[G] \otimes A \longrightarrow J_G \otimes A \longrightarrow 0$$

Proof. Recall that we have exact sequences

$$0 \longrightarrow I_G \longrightarrow \mathbb{Z}[G] \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow A \longrightarrow \mathbb{Z}[G] \longrightarrow J_G \longrightarrow 0$$

By Proposition 1.2.6, all groups involved are free $\mathbb{Z}[G]$-modules. Appealing to Proposition 1.4.3 yields the desired exact sequences. ☐

3.3 Dimension Shifting

Theorem 3.3.1 (Dimension Shifting)**.** *Let A be a G-module and $H \subseteq G$ a subgroup. Define the G-modules*

$$\begin{aligned} A^m &= \underbrace{J_G \otimes \cdots \otimes J_G}_{m \text{ times}} \otimes A \\ A^{-m} &= \underbrace{I_G \otimes \cdots \otimes I_G}_{m \text{ times}} \otimes A \end{aligned}$$

Then the m-fold composition of the connecting homomorphism δ induces an isomorphism

$$\delta^m : H^{q-m}(H, A^m) \to H^q(H, A)$$

Proof. Since $\mathbb{Z}[G] \otimes A$ is cohomologically trivial, applying the functor $H^q(H, -)$ to the exact sequences of Lemma 3.2.7 yields isomorphisms

$$\delta : H^{q-1}(H, A) \cong H^q(H, I_G \otimes A)$$
$$\delta : H^{q-1}(H, J_G \otimes A) \cong H^q(H, A)$$

Iterating this process yields isomorphisms for all $m \in \mathbb{Z}$. □

Corollary 3.3.2. *Let A be a G-module. Then for all $q \in \mathbb{Z}$, $H^q(G, A)$ is torsion. In particular, the order of the elements of $H^q(G, A)$ divide $|G|$.*

Proof. First suppose that $q = 0$. Recall that $H^0(G, A) = A^G/N_G A$. Let $n = |G|$ and $a \in A^G$. Then $N_G a = na$ whence $n \cdot H^q(G, A) = 0$. The general case for all q then follows via dimension shifting. □

Corollary 3.3.3. *Let A be a uniquely divisible G-module. Then A has trivial cohomology.*

Proof. Since A is uniquely divisible, the multiplication-by-n map $n : A \to A$ is a bijection for all $n \geq 1$. This induces an isomorphism of cohomology groups $n : H^q(H, A) \to H^q(H, A)$ for all subgroups $H \subseteq G$. In particular, if $n = |G|$ then we have

$$H^q(H, A) = n \cdot H^q(H, A) = 0$$

by Corollary 3.3.2. □

Corollary 3.3.4. *Consider $\mathbb{Z}$ and $\mathbb{Q}$ as G-modules with the trivial action. Then $H^2(G, \mathbb{Z}) \cong \chi(G)$ where $\chi(G)$ is the character group of G.*

Proof. We first observe that we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Now, $\mathbb{Q}$ is uniquely divisible and so Corollary 3.3.3 implies that $\mathbb{Q}$ is cohomologically trivial. We then have

$$H^2(G, \mathbb{Z}) \cong H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) = \chi(G)$$

as required. □

Definition 3.3.5. Let G be a group and $g, h \in G$. The **commutator** of g and h is defined to be

$$[g, h] = g^{-1}h^{-1}gh$$

An element of G that is of the form $[g, h]$ for some $g, h \in G$ is called a **commutator**. We define the **commutator subgroup** $[G, G]$ of G to be the one generated by the commutators of G. We define the **abelianisation** of G, denoted G^{ab} to be $G/[G, G]$.

Remark.

1. Consider the inclusion functor i : **AbGrp** → **Grp**. Then the functor F : **Grp** → **AbGrp** is a left-adjoint for i.
2. It is immediately clear that G is abelian if and only if it is equal to its abelianisation.

Theorem 3.3.6. *Consider $\mathbb{Z}$ as a G-module with the trivial action. Then $H^{-2}(G, \mathbb{Z}) \cong G^{\mathrm{ab}}$.*

Proof. Since $\mathbb{Z}[G]$ is G-induced, it has trivial cohomology. Applying the functor $H^q(G,-)$ to the exact sequence

$$0 \longrightarrow I_G \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

yields an isomorphism $H^{-2}(G,\mathbb{Z}) \cong H^{-1}(G,I_G)$. Now, by definition, we have $H^{-1}(G,I_G) = I_G/I_G^2$ so we need to exhibit an isomorphism $G/[G,G] \cong I_G/I_G^2$. We claim that the map

$$\begin{aligned} \log : G &\to I_G/I_G^2 \\ \sigma &\mapsto (\sigma - 1) + I_G^2 \end{aligned}$$

induces such an isomorphism. We must first check that ϕ is indeed a homomorphism. To this end, fix $\sigma, \tau \in G$. Then

$$\begin{aligned} \phi(\sigma\tau) = (\sigma\tau - 1) + I_G^2 &= [(\sigma - 1) + (\tau - 1) + (\sigma - 1)(\tau - 1)] + I_G^2 \\ &= [(\sigma - 1) + I_G^2] + [(\tau - 1) + I_G^2] = \phi(\sigma)\phi(\tau) \end{aligned}$$

Now, I_G/I_G^2 is abelian so ker(log) necessarily contains the commutator subgroup of G. We then have an induced homomorphism

$$\log : G/[G,G] \to I_G/I_G^2$$

which we claim is an isomorphism. In order to show that log is bijective, we shall construct it's inverse. Recall that I_G is the free abelian group on $\sigma - 1$ for $\sigma \in G\setminus\{1\}$. Then the map

$$\exp : I_G \to G/[G,G]$$

given by $(\sigma - 1) \mapsto \sigma[G,G]$ is clearly a surjective homomorphism. Now, given $\sigma, \tau \in G$ with $\sigma, \tau \neq 1$ we have

$$(\sigma - 1) \cdot (\tau - 1) = (\sigma\tau - 1) - (\sigma - 1) - (\tau - 1) \mapsto \sigma\tau\sigma^{-1}\tau^{-1}[G,G] = 1$$

and so the elements of I_G^2 are in ker(exp). We then have an induced homomorphism

$$\exp : I_G/I_G^2 \to G/[G,G]$$

satisfying $\exp \circ \log = \mathrm{id}$ and $\log \circ \exp = \mathrm{id}$ whence $\log : G^{\mathrm{ab}} \cong I_G/I_G^2$ is an isomorphism. □

4 Inflation, Restriction and Corestriction

Throughout this section, G will always be a finite group and $H \subseteq G$ a subgroup.

4.1 Inflation and Restriction

Unless otherwise stated, q shall refer to an element of $\mathbb{Z}_{\geq 1}$.

Definition 4.1.1. Suppose that H is normal in G. We define the ***q*-inflation** map to be the map

$$\inf_q : \{ (G/H)^q \to A^H \} \to \{ G^q \to A \}$$

defined as follows. Given a q-cochain $x : (G/H)^q \to A^H$, define $y = \inf_q(x)$ to be the q-cochain

$$y(\sigma_1, \ldots, \sigma_q) = x(\sigma_1 H, \ldots, \sigma_q H)$$

Proposition 4.1.2. *Suppose that H is normal in G. Then the q-inflation map satisfies*

$$\inf_{q+1} \circ \partial_{q+1} = \partial_{q+1} \circ \inf_q$$

Hence the q-inflation map descends to a homomorphism

$$\inf_q : H^q(G/H, A^H) \to H^q(G, A)$$

Proof. Fix a q-cochain $x : (G/H)^q \to A^H$. If $q > 1$ we then have that

$$\begin{aligned}
(\inf_{q+1} \circ \partial_{q+1})(x)(\sigma_1, \ldots, \sigma_{q+1}) &= \partial_{q+1}(x)(\sigma_1 H, \ldots, \sigma_{q+1} H) \\
&= (x \circ d_{q+1})(\sigma_1 H, \ldots, \sigma_{q+1} H) \\
&= x(\sigma_2 H, \ldots, \sigma_{q+1} H)^{\sigma_1} \\
&+ \sum_{i=1}^{q} (-1)^i x(\sigma_1 H, \ldots, \sigma_{i-1} H, \sigma_i \sigma_{i+1} H, \sigma_{i+2} H, \ldots, \sigma_{q+1} H) \\
&+ (-1)^{q+1} x(\sigma_1 H, \ldots, \sigma_q H) \\
&= \inf_q(x)(\sigma_2, \ldots, \sigma_{q+1})^{\sigma_1} \\
&+ \sum_{i=1}^{q} (-1)^i \inf_q(x)(\sigma_1, \ldots, \sigma_{i-1}, \sigma_i \sigma_{i+1}, \sigma_{i+1}, \ldots, \sigma_{q+1}) \\
&+ (-1)^{q+1} \inf_q(x)(\sigma_1, \ldots, \sigma_q) \\
&= (\partial_{q+1} \circ \inf_q)(x)(\sigma_1, \ldots, \sigma_{q+1})
\end{aligned}$$

If $q = 1$ then the proof is immediate. It then follows that $\inf_q$ sends cocycles to cocycles and coboundaries to coboundaries so we get an induced homomorphism of cohomology groups. □

Definition 4.1.3. We define the ***q*-restriction** map to be the map

$$\operatorname{res}_q : \{ G^q \to A \} \to \{ H^q \to A \}$$

defined as follows. Given a q-cochain $x : G^q \to A$, define $\operatorname{res}_q(x)$ to be the q-cochain $H^q \to A$ given by restricting x to H^q.

Proposition 4.1.4. *The q-restriction map satisfies*

$$\operatorname{res}_{q+1} \circ \partial_{q+1} = \partial_{q+1} \circ \operatorname{res}_q$$

Proof. This is proved in the same way as for the inflation map. □

Proposition 4.1.5. *Suppose that H is normal in G and $f : A \to B$ is a homomorphism of G-modules. Then the diagrams*

$$\begin{array}{ccc}
H^q(G/H, A^H) & \xrightarrow{\overline{f}} & H^q(G/H, B^H) \\
\downarrow{\scriptstyle \inf_q} & & \downarrow{\scriptstyle \inf_q} \\
H^q(G, A) & \xrightarrow{\overline{f}} & H^q(G, B)
\end{array}
\qquad
\begin{array}{ccc}
H^q(G, A) & \xrightarrow{\overline{f}} & H^q(G, B) \\
\downarrow{\scriptstyle \operatorname{res}_q} & & \downarrow{\scriptstyle \operatorname{res}_q} \\
H^q(H, A) & \xrightarrow{\overline{f}} & H^q(H, B)
\end{array}$$

commute. Note that the normality condition is not needed in the second diagram.

Proof. We prove the Proposition for the restriction diagram. The one for inflation follows from a similar argument. Fix $[c] \in H^q(G, A)$. Then

$$(\text{res}_q \circ \overline{f})([c]) = \text{res}_q([f \circ c]) = f|H \circ c|_H = \overline{f}([c|_H]) = (\overline{f} \circ \text{res}_q)([c])$$

□

Proposition 4.1.6. *Let*

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0 \tag{2}$$

be an exact sequence in $\boldsymbol{G}_{\textbf{mod}}$. *Suppose that* H *is normal in* G *and that the sequence*

$$0 \longrightarrow A^H \xrightarrow{\phi} B^H \xrightarrow{\psi} C^H \longrightarrow 0 \tag{3}$$

is exact. Then the diagram

$$\begin{array}{ccc} H^q(G/H, C^H) & \xrightarrow{\delta_q} & H^{q+1}(G/H, A^H) \\ \big\downarrow \text{inf}_q & & \big\downarrow \text{inf}_{q+1} \\ H^q(G, C) & \xrightarrow{\delta_q} & H^{q+1}(G, A) \end{array}$$

commutes.

Proof. Fix a cohomology class $\overline{c'_q} \in H^q(G/H, C^H)$. By exactness of Sequence 3, there exists $b'_q \in B^H$ such that $\psi_q(b'_q) = c'_q$. Moreover, there exists $a'_{q+1} \in A^H$ such that $\phi_{q+1}(a'_{q+1}) = \partial_q(b'_q)$. Then $\delta_q(\overline{c'_q}) = \overline{a'_{q+1}}$.

Conversely, by exactness of Sequence 2, there exists $b_q \in B$ such that $\psi_q(b_q) = \text{inf}_q(c'_q)$. Moreover, there exists $a_{q+1} \in A$ such that $\phi_{q+1}(a_{q+1}) = \partial_q(b_q)$. Then $\delta_q(\text{inf}_q(\overline{c'_q})) = \overline{a_{q+1}}$.

Now,

$$\begin{aligned}(\phi_{q+1} \circ \text{inf}_{q+1})(a'_{q+1}) &= (\text{inf}_{q+1} \circ \phi_q)(a'_{q+1}) \\ &= (\text{inf}_{q+1} \circ \partial_q)(b'_q) \\ &= (\partial_q \circ \text{inf}_q)(b'_q)\end{aligned}$$

But observe that

$$\psi_q(b_q) = \text{inf}_q(c'_q) = (\text{inf}_q \circ \psi_q)(b'_q) = (\psi_q \circ \text{inf}_q)(b'_q) = \psi_q(\text{inf}_q(b'_q))$$

and so $\text{inf}_q(b'_q)$ is a preimage of $\text{inf}_q(c'_q)$. But the definition of a'_{q+1} is independent of the choice of such a preimage so we then have that

$$\begin{aligned}(\phi_{q+1} \circ \text{inf}_{q+1})(a'_{q+1}) &= \partial_q(b_q) \\ &= \phi_{q+1}(a_{q+1})\end{aligned}$$

But ϕ is injective and so $\text{inf}_{q+1}(a'_{q+1}) = a_{q+1}$ whence $\text{inf}_{q+1} \circ \delta_q = \delta_q \circ \text{inf}_q$. □

Proposition 4.1.7. *Let*

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

be an exact sequence in $\boldsymbol{G}_{\mathbf{mod}}$. Then the diagram

$$\begin{array}{ccc} H^q(G,C) & \xrightarrow{\delta_q} & H^{q+1}(G,A) \\ \Big\downarrow{\scriptstyle \text{res}_q} & & \Big\downarrow{\scriptstyle \text{res}_{q+1}} \\ H^q(H,C) & \xrightarrow{\delta_q} & H^{q+1}(H,A) \end{array}$$

commutes.

Proof. This follows the same reasoning as the proof for the previous Proposition. □

Theorem 4.1.8. *Let A be a G-module and suppose that H is normal in G. Then the sequence*

$$0 \longrightarrow H^1(G/H, A^H) \xrightarrow{\text{inf}} H^1(G,A) \xrightarrow{\text{res}} H^1(H,A)$$

is exact.

Proof. We first prove that the inflation map is injective. To this end, fix a 1-cocycle $x : G/H \to A^H$ such that $\inf(x)$ is a 1-coboundary with respect to A. Then

$$\inf(x)(\sigma) = x(\sigma H) = a^\sigma - a \qquad \text{(for some } a \in A)$$

This implies that for all $\tau \in H$ we have

$$a^{\sigma\tau} - a = a^\sigma - a$$

whence $a^\tau = a$. Hence $a \in A^H$ and so x is a 1-coboundary $x(\sigma H) = a^{\sigma H} - a$.

We must now show exactness at $H^1(G,A)$. In other words, we need to show that $\ker(\text{res}) = \text{im}(\text{inf})$. To this end, fix a 1-cocycle $x : G/H \to A^H$. Given $\sigma \in H$ we have

$$(\text{res} \circ \inf)(x)(\sigma) = \inf(x)(\sigma) = x(\sigma H) = x(1)$$

Now, since x is a 1-cocycle, we have that

$$x(1) = x(1 \cdot 1) = x(1)^1 + x(1) = x(1) + x(1) = 0$$

We thus see that $\text{im}(\text{inf}) \subseteq \ker(\text{res})$. Conversely, suppose that $x \in \ker(\text{res})$. Then $x : G \to A$ is a 1-cocycle that restricts to a 1-coboundary of the H-module A:

$$x(\tau) = a^\tau - a \qquad \text{(for all } \tau \in H, \text{ some } a \in A)$$

Now let $\rho : G \to A$ be the 1-coboundary given by $\rho(\sigma) = a^\sigma - a$. Then the 1-cocycle $x'(\sigma) = x(\sigma) - \rho(\sigma)$ is in the same cohomology class as x and restricts to the zero map on H.

Now, define $y : G/H \to A$ by $y(\sigma H) = x'(\sigma)$. We claim that y is a 1-cocycle with respect to A^G. We must first check that it is well-defined. To this end, suppose that $\tau H = \sigma H$. Then $\tau = \sigma\pi$ for some $\pi \in H$. Then

$$y(\tau H) = y(\sigma\pi H) = x'(\sigma\pi) = x'(\sigma) + x'(\pi)^\sigma = x'(\sigma) = y(\sigma H)$$

Furthermore,

$$x(\tau\sigma) = x(\tau) + x(\sigma)^\tau = x(\sigma)^\tau \qquad \text{(for all } \tau \in G)$$

Since $y(\sigma H) = y(\tau\sigma H)$ for all $\tau \in H$, it then follows that

$$y(\sigma H) = y(\tau\sigma H) = x(\sigma)^\tau = y(\sigma H)^\tau \qquad \text{(for all } \tau \in H)$$

so that $y(\sigma H) \in A^G$ and so y is in fact a 1-cocyle with respect to A^G. It is clear that $\inf(y) = x'$. Modding out by coboundaries, we then see that $\ker(\text{res}) \subseteq \text{im}(\inf)$. ☐

Theorem 4.1.9. *Let A be a G-module and suppose that H is normal in G. If $H^i(H, A) = 0$ for all $1 \leq i \leq q-1$ then the sequence*

$$0 \longrightarrow H^q(G/H, A^H) \xrightarrow{\inf_q} H^q(G, A) \xrightarrow{\text{res}_q} H^q(H, A)$$

is exact.

Proof. We prove the Theorem by induction on the dimension q by dimension shifting. Theorem 4.1.8 provides the basis case for the induction. Now set $B = \mathbb{Z}[G] \otimes A$ and $C = J_G \otimes A$. Then we have an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

By hypothesis, we have that $H^1(H, A) = 0$, so Proposition 3.1.4 yields an exact sequence

$$0 \longrightarrow A^H \longrightarrow B^H \longrightarrow C^H \longrightarrow 0$$

By Proposition 4.1.6 we then have a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H^{q-1}(G/H, C^H) & \xrightarrow{\inf_{q-1}} & H^{q-1}(G, C) & \xrightarrow{\text{res}_{q-1}} & H^{q-1}(H, C) \\ & \downarrow \delta_{q-1} & & \downarrow \delta_{q-1} & & \downarrow \delta_{q-1} \\ 0 \longrightarrow & H^q(G/H, A^H) & \xrightarrow{\inf_q} & H^q(G, A) & \xrightarrow{\text{res}_q} & H^q(H, A) \end{array}$$

Now, B is G-induced and H-induced and B^H is G/H-induced so that the connecting homomorphisms δ_{q-1} are all isomorphisms. By the induction hypothesis, the first row is exact so we must have that the second row is also exact. ☐

Definition 4.1.10. We define the **0-restriction** map to be the map

$$\begin{aligned} \text{res}_0 : H^0(G, A) &\to H^0(H, A) \\ a + N_G A &\mapsto a + N_H A \end{aligned}$$

Lemma 4.1.11. *Let*

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

be an exact sequence in $\boldsymbol{G}_{\textbf{mod}}$. Then the diagram

$$\begin{array}{ccc} H^0(G, C) & \xrightarrow{\delta_0} & H^1(G, A) \\ \downarrow \text{res}_0 & & \downarrow \text{res}_1 \\ H^0(H, C) & \xrightarrow{\delta_0} & H^1(H, A) \end{array}$$

commutes.

Proof. Let $c \in C^G$ be a 0-cocycle and $\overline{c} = c + N_G C$ it's image in $H^0(G, C)$. Then $\mathrm{res}_0(\overline{c}) = c + N_H C$ so that c is also a 0-cocyle of the H-module C. Let $b \in B$ be such that $\psi(b) = c$ and $a_1 : G \to A$ a 1-cocycle such that $\phi_1(a_1) = \partial_0(b)$. Then $\delta_0(\overline{c}) = \overline{a_1}$ and

$$(\delta_0 \circ \mathrm{res}_0)(\overline{c}) = \overline{\mathrm{res}_1 a_1} = \mathrm{res}_1 \overline{a_1} = (\mathrm{res}_1 \circ \delta_0)(\overline{c})$$

□

Theorem 4.1.12. *Let $q \in \mathbb{Z}$. Then **restriction** is the family of homomorphisms*

$$\mathrm{res}_q : H^q(G, A) \to H^q(H, A)$$

uniquely determined by the properties

1. *When $q = 0$ we explicitly have*

$$\begin{aligned} \mathrm{res}_0 : H^0(G, A) &\to H^0(H, A) \\ a + N_G A &\mapsto a + N_H A \end{aligned}$$

2. *Given an exact sequence*

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

we have a commutative diagram

$$\begin{array}{ccc} H^q(G, C) & \xrightarrow{\delta_q} & H^{q+1}(G, C) \\ \downarrow \mathrm{res}_q & & \downarrow \mathrm{res}_{q+1} \\ H^q(H, C) & \xrightarrow{\delta_q} & H^{q+1}(H, C) \end{array}$$

Proof. Via dimension shifting, the q-fold composition of δ provides us with isomorphisms $\delta_q : H^0(G, A^q) \to H^q(G, A)$ and $\delta_q : H^0(H, A^q) \to H^q(H, A)$ that fit into the diagram

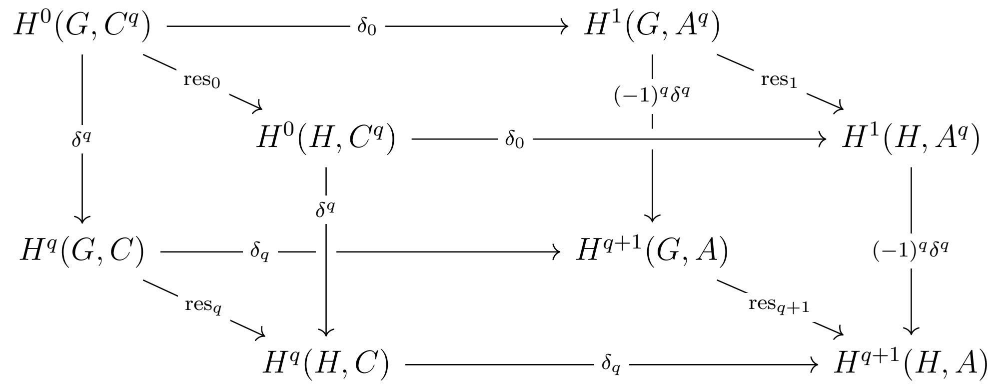
$$\begin{array}{ccc} H^0(G, A^q) & \xrightarrow{\delta^q} & H^q(G, A) \\ \downarrow \mathrm{res}_0 & & \vdots\, \mathrm{res}_q \\ H^0(H, A^q) & \xrightarrow{\delta^q} & H^q(H, A) \end{array}$$

We define res_q to be the homomorphism extending the above diagram. Then it is clear that the res_q are unique and coincide with the previous definition of res_q. It remains to verify that res_q satisfies the second property in the Theorem.

By induction, we obtain an exact sequence

$$0 \longrightarrow A^q \longrightarrow B^q \longrightarrow C^q \longrightarrow 0$$

Now consider the diagram



The commutativity of the top square is guaranteed by Lemma 4.1.11. The commutativity of the back and front squares are guaranteed by q applications of Proposition 3.1.6. The commutativity of the side squares is guaranteed by the definition of res_q. This then implies that the bottom square is also commutative since the maps transferring from the top square to the bottom square are all isomorphisms. □

Definition 4.1.13. Let A be a G-module. We define the **Verlagerung** or **transfer** from G to H to be the homomorphism

$$\mathrm{Ver} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$$

induced by the restriction $H^{-2}(G, \mathbb{Z}) \to H^{-2}(H, \mathbb{Z})$.

4.2 Corestriction

Definition 4.2.1. We define the **(-1)-corestriction** map to be the homomorphism

$$\begin{aligned} \mathrm{cores}_{-1} : H^{-1}(H, A) &\to H^{-1}(G, A) \\ a + I_H A &\mapsto a + I_G A \end{aligned}$$

Similarly, we define the **0-corestriction** map to be the homomorphism

$$\begin{aligned} \mathrm{cores}_0 : H^0(H, A) &\to H^0(G, A) \\ a + N_H A &\mapsto N_{G/H} a + N_G A \end{aligned}$$

where $N_{G/H} a = \sum_{\sigma_i} a^{\sigma_i} \in A^G$ where the σ_i are a set of left coset representatives of H in G.

Lemma 4.2.2. *Let*

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

be an exact sequence in $\boldsymbol{G}_{\mathbf{mod}}$. Then the diagram

$$\begin{array}{ccc} H^{-1}(H, C) & \xrightarrow{\delta_{-1}} & H^0(H, A) \\ \big\downarrow \mathrm{cores}_{-1} & & \big\downarrow \mathrm{cores}_0 \\ H^{-1}(G, C) & \xrightarrow{\delta_{-1}} & H^0(G, A) \end{array}$$

commutes.

Proof. Fix a (-1)-cocyle $c \in {}_{N_H}C$ and let $\overline{c} = c + I_H C$ be the corresponding cohomology class. Then $c \in {}_{N_G}C$ is a representative of the cohomology class $\text{cores}_{-1}(\overline{c}) = c + I_G C \in H^{-1}(G, C)$. Choose $b \in B$ such that $\psi(b) = c$ and $a_0 \in A$ such that $\phi(a_0) = \partial_0(b) = N_H b$ so that

$$(\text{cores}_0 \circ \delta_{-1})(\overline{c}) = N_{G/H} a_0 + N_G A$$

On the other hand, we have

$$\partial_0(b) = N_G b = N_{G/H} N_H b = N_{G/H}(\phi(a_0)) = \phi(N_{G/H} a_0)$$

hence $\delta_{-1}(c + I_G C) = N_{G/H} a_0 + N_G A$ whence

$$(\delta_{-1} \circ \text{cores}_{-1})(\overline{c}) = N_{G/H} a_0 + N_G A = (\text{cores}_0 \circ \delta_{-1})(\overline{c})$$

□

Theorem 4.2.3. *Let $q \in \mathbb{Z}$. Then* ***corestriction*** *is the family of homomorphisms*

$$\text{cores}_q : H^q(H, A) \to H^q(G, A)$$

uniquely determined by the properties

1. *When $q = 0$ we explicitly have*

$$\begin{aligned} \text{res}_0 : H^0(H, A) &\to H^0(G, A) \\ a + N_H A &\mapsto N_{G/H} a + N_G A \end{aligned}$$

2. *Given an exact sequence*

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

we have a commutative diagram

$$\begin{array}{ccc} H^q(H, C) & \xrightarrow{\delta_q} & H^{q+1}(H, C) \\ \downarrow{\scriptstyle \text{cores}_q} & & \downarrow{\scriptstyle \text{cores}_{q+1}} \\ H^q(G, C) & \xrightarrow{\delta_q} & H^{q+1}(G, C) \end{array}$$

Proof. The proof is dual to that for restriction. □

Theorem 4.2.4. *The homomorphism*

$$\kappa : H^{\text{ab}} \to G^{\text{ab}}$$

induced by the corestriction

$$\text{cores}_{-2} : H^{-2}(H, \mathbb{Z}) \to H^{-2}(G, \mathbb{Z})$$

coincides with the canonical homomorphism $\sigma[H, H] \mapsto \sigma[G, G]$.

Proof. This follows immediately from the commutative diagram

$$\begin{array}{ccccc}
H^{-2}(H,\mathbb{Z}) & \xrightarrow[\sim]{\delta_{-2}} & H^{-1}(H,I_H) = I_H/I_H^2 & \xleftarrow[\sim]{\log} & H^{\mathrm{ab}} \\
\Big\downarrow \mathrm{cores}_{-2} & & \Big\downarrow \mathrm{cores}_{-1} & & \Big\downarrow \kappa \\
H^{-2}(G,\mathbb{Z}) & \xrightarrow[\sim]{\delta_{-2}} & H^{-1}(G,I_G) = I_G/I_G^2 & \xleftarrow[\sim]{\log} & G^{\mathrm{ab}}
\end{array}$$

obtained via dimension shifting. Here the map cores_{-1} is understood to be the composition of the natural map $H^{-1}(H, I_H) \to H^{-1}(H, I_G)$ with $\mathrm{cores}_{-1} : H^{-1}(H, I_G) \to H^{-1}(G, I_G)$. □

Theorem 4.2.5. *Let A be a G-module. Then the composition*

$$H^q(G,A) \xrightarrow{\mathrm{res}_q} H^q(H,A) \xrightarrow{\mathrm{cores}_q} H^q(G,A)$$

is the endomorphism

$$\mathrm{cores}_q \circ \mathrm{res}_q = [G : H] \cdot \mathrm{id}$$

Proof. First suppose that $q = 0$ and fix a cohomology class $\overline{a} = a + N_G A \in H^q(G, A)$ for some $a \in A^G$. Then

$$(\mathrm{cores}_0 \circ \mathrm{res}_0)(\overline{a}) = \mathrm{cores}_0(a + N_H A) = N_{G/H} a + N_G A = [G : H]a + N_G A = [G : H] \cdot \overline{a}$$

Via dimension shifting, we have the commutative diagram

$$\begin{array}{ccc}
H^0(G,A^q) & \xrightarrow{\mathrm{cores}_0 \circ \mathrm{res}_0} & H^0(G,A^q) \\
\Big\downarrow \delta^q & & \Big\downarrow \delta_q \\
H^q(G,A) & \xrightarrow{\mathrm{cores}_q \circ \mathrm{res}_q} & H^q(G,A)
\end{array}$$

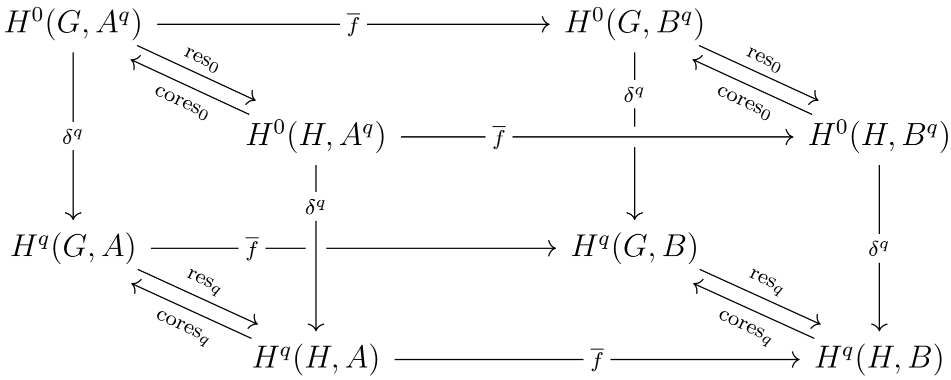
Since the vertical maps are isomorphisms and the top map is multiplication by $[G : H]$, it follows that the bottom map must also be multiplication by $[G : H]$. □

Proposition 4.2.6. *Let $f : A \to B$ be a homomorphism of G-modules. Then the diagram*

$$\begin{array}{ccc}
H^q(G,A) & \xrightarrow{\overline{f}} & H^q(G,B) \\
\mathrm{cores}_q \Big\uparrow \Big\downarrow \mathrm{res}_q & & \mathrm{cores}_q \Big\uparrow \Big\downarrow \mathrm{res}_q \\
H^q(H,A) & \xrightarrow{\overline{f}} & H^q(H,B)
\end{array}$$

commutes.

Proof. This follows immediately from the definitions in the case that $q = 0$. For the general case first note that the homomorphism $f : A \to B$ induces a homomorphism $f : A^q \to B^q$. Now consider the diagram



The back and front squares are commutative by Proposition 3.1.5. The side squares are commutative by Theorems 4.1.12 and 4.2.3. The top square is commutative from the case when $q = 0$. Since the vertical maps are all isomorphisms, it follows that the bottom square must also be commutative. □

Remark. Let A be a torsion abelian group. By the Chinese Remainder Theorem, A admits a decomposition into its p-Sylow subgroups A_p where A_p consists of all elements of A of p-power order. We refer to A_p as the ***p*-primary part** of A.

Proposition 4.2.7. *Let A be a G-module and G_p a p-Sylow subgroup of G. Then for all $q \in \mathbb{Z}$ we have*

$$\text{res}_q : H^q(G, A)_p \to H^q(G_p, A)$$

is injective and

$$\text{cores}_q : H^q(G_p, A) \to H^q(G, A)_p$$

is surjective.

Proof. We have that $\text{cores}_q \circ \text{res}_q = [G : G_p] \cdot \text{id}$. But $[G : G_p]$ and p are relatively prime so that $\text{cores}_q \circ \text{res}_q$ is an automorphism of $H^q(G, A)_p$. Hence if we suppose that for $x \in H^q(G, A)_p$ we have that $\text{res}_q(x) = 0$, it then follows that $\text{cores}_q \circ \text{res}_q(x) = 0$ whence $x = 0$ and so res_q is inejctive.

To see the second claim we note that by Corollary 3.3.2, the elements of $H^q(G_p, A)$ have p-power order whence $\text{cores}_q(H^q(G_p, A)) \subseteq H^q(G, A)_p$. But $\text{cores}_q \circ \text{res}_q$ is a bijection on $H^q(G, A)_p$ so we must have that $\text{im}(\text{cores}_q) = H^q(G, A)_p$. □

Corollary 4.2.8. *Let A be a G-module. Suppose that for every prime p there exists a p-Sylow subgroup G_p of G such that $H^q(G_p, A) = 0$. Then $H^q(G, A) = 0$.*

Proof. By Proposition 4.2.7, we have an injection

$$\text{res}_q : H^q(G, A)_p \to H^q(G_p, A)$$

By hypothesis, each such $H^q(G_p, A) = 0$ whence $H^q(G, A)_p = 0$ for every prime p. But $H^q(G, A)$ is torsion and is thus the direct sum of its p-Sylow subgroups. Hence $H^q(G, A) = 0$. □

4.3 G/H-induced Modules

Definition 4.3.1. Let A be a G-module. We say that A is ***G/H*-induced** if

$$A = \bigoplus_{\sigma \in G/H} D^\sigma$$

for some H-module D and $\sigma \in G/H$ runs over a set of left coset representatives of H in G.

Theorem 4.3.2 (Shapiro's Lemma)**.** *Let $A = \bigoplus_{\sigma \in G/H} D^\sigma$ be a G/H-induced module. Then*

$$H^q(G, A) \cong H^q(H, D)$$

via the composition

$$H^q(G,A) \xrightarrow{\mathrm{res}_q} H^q(H,A) \xrightarrow{\overline{\pi}} H^q(H,D)$$

where $\overline{\pi}$ is induced by the canonical projection $\pi : A \to D$.

Proof. First suppose that $q = 0$. Let $\{\sigma_i\}_{1\le i\le m}$ (with $\sigma_1 = 1$) be a set of left coset representatives of H in G so that $A = \bigoplus_{i=1}^m D^{\sigma_i}$. We define a map, which we claim is the inverse of the composition

$$A^G/N_GA \xrightarrow{\mathrm{res}_0} A^H/N_HA \xrightarrow{\overline{\pi}} D^H/N_HD$$

by

$$\begin{aligned} \nu : D^H/N_HD &\to A^G/N_GA \\ d + N_HD &\mapsto \left(\sum_{i=1}^m d^{\sigma_i}\right) + N_GA \end{aligned}$$

We must first check that this definition is well-defined. Suppose that $d+N_HD = d'+N_HD$. Then $d' = d + z$ for some $z = \sum_{\tau\in H} z_\tau$ with $\tau \in D$. It then follows that

$$\nu(d' + N_HD) = \left(\sum_{i=1}^m d^{\sigma_i} + \sum_{i=1}^m \sum_{\tau\in H} z^{\sigma_i\tau}\right) + N_GA = d + N_GA = \nu(d + N_HD)$$

To see that ν is the inverse of $\overline{\pi}\circ\mathrm{res}_0$, first fix $a + N_GA \in A^G/N_GA$. Then

$$\begin{aligned} (\nu\circ\overline{\pi}\circ\mathrm{res}_0)(a+N_GA) = (\nu\circ\overline{\pi})(a+N_HA) &= \nu(\pi(a)+N_HD) \\ &= \left(\sum_{i=1}^m \pi(a)^{\sigma_i}\right) + N_GA \\ &= a + N_GA \end{aligned}$$

The composition in the opposite direction follows from a similar argument. For the general case set for all $q \ge 0$

$$\begin{aligned} A^q &= J_G\otimes\cdots\otimes J_G\otimes A, & A^{-q} &= I_G\otimes\cdots\otimes I_G\otimes A \\ D_G^q &= J_G\otimes\cdots\otimes J_G\otimes D, & D_G^{-q} &= I_G\otimes\cdots\otimes I_G\otimes D \\ D_H^q &= J_H\otimes\cdots\otimes J_H\otimes D, & D_H^{-q} &= I_H\otimes\cdots\otimes I_H\otimes D \end{aligned}$$

Observe that by Proposition 1.2.6 we have

$$\begin{aligned} I_G &= I_H \oplus \bigoplus_{\tau\in G}\left(\sum_{i=2}^m \mathbb{Z}(\sigma_i^{-1}-1)\right)^\tau \\ J_G &= J_H \oplus \bigoplus_{\tau\in G}\left(\sum_{i=2}^m \mathbb{Z}\overline{\sigma_i}^{-1}\right)^\tau \end{aligned}$$

so that $D_G^q = D_H^q \oplus C^q$ for some H-induced H-module C^q. Dimension shifting then provides us with a commutative diagram

$$\begin{array}{ccccccc} H^0(G,A^q) & \xrightarrow{\mathrm{res}_0} & H^0(H,A^q) & \xrightarrow{\overline{\pi_H}} & H^0(H,D_H^q) & \xrightarrow{\overline{\rho}} & H^0(H,D_G^q) \\ \downarrow{\scriptstyle\delta^q} & & \downarrow{\scriptstyle\delta_q} & & & & \downarrow{\scriptstyle\delta_q} \\ H^q(G,A) & \xrightarrow{\mathrm{res}_q} & H^q(H,A) & & \xrightarrow{\overline{\pi}} & & H^q(H,D) \end{array}$$

The composite $\overline{\pi_H} \circ \mathrm{res}_0$ is bijective by the special case in 0-dimensions. $\overline{\rho}$ is bijective because of the coadditivity of $H^0(H,-)$ and the fact that H-induced modules have trivial cohomology. Moreover, it is clear that the composition $\rho \circ \pi_H$ is induced by the projection $\pi : A \to D$ so that the right-hand square commutes. Since the vertical maps are all isomorphisms, it follows that $\overline{\pi} \circ \mathrm{res}_q$ is also an isomorphism. □

5 The Cup Product

5.1 Definition

Theorem 5.1.1. *Let A, B be G-modules and $p, q \in \mathbb{Z}$. Then there exists a family of maps*

$$\smile : H^p(G, A) \times H^q(G, B) \to H^{p+q}(G, A \otimes B)$$

*called the **cup product** which is uniquely determined by the conditions*

1. *When $q = p = 0$ we have*

$$\begin{aligned} \smile : H^0(G, A) \times H^0(G, B) &\to H^0(G, A \otimes B) \\ (\overline{a}, \overline{b}) &\mapsto \overline{a \otimes b} \end{aligned}$$

2. *Given exact sequences*

$$0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$$

$$0 \longrightarrow A \otimes B \longrightarrow A' \otimes B \longrightarrow A'' \otimes B \longrightarrow 0$$

in $\boldsymbol{G}_{\mathbf{mod}}$, we have a commutative diagram

$$\begin{array}{ccccc} H^p(G, A'') & \times & H^q(G, B) & \xrightarrow{\smile} & H^{p+q}(G, A'' \otimes B) \\ \downarrow \delta_p & & \downarrow \mathrm{id} & & \downarrow \delta_{p+q} \\ H^{p+1}(G, A) & \times & H^q(G, B) & \xrightarrow{\smile} & H^{p+q+1}(G, A \otimes B) \end{array}$$

so that $\delta_{p+q}(\overline{a''} \smile \overline{b}) = \delta_p(\overline{a''}) \smile \overline{b}$ for $\overline{a''} \in H^p(G, A'')$ and $\overline{b} \in H^q(G, B)$.

3. *Given exact sequences*

$$0 \longrightarrow B \longrightarrow B' \longrightarrow B'' \longrightarrow 0$$

$$0 \longrightarrow A \otimes B \longrightarrow A \otimes B' \longrightarrow A \otimes B'' \longrightarrow 0$$

in $\boldsymbol{G}_{\mathbf{mod}}$, we have a commutative diagram

$$\begin{array}{ccccc} H^p(G, A) & \times & H^q(G, B'') & \xrightarrow{\smile} & H^{p+q}(G, A \otimes B'') \\ \downarrow \mathrm{id} & & \downarrow \delta_q & & \downarrow (-1)^q \delta_{p+q} \\ H^p(G, A) & \times & H^{q+1}(G, B) & \xrightarrow{\smile} & H^{p+q+1}(G, A \otimes B) \end{array}$$

so that $\delta_{p+q}(\overline{a} \smile \overline{b''}) = (-1)^q(\overline{a} \smile \delta_q(\overline{b''}))$ *for* $\overline{a} \in H^p(G, A)$ *and* $\overline{b''} \in H^q(G, B'')$.

Proof. We first note that in dimensions $p = q = 0$, the cup product is indeed well-defined since we have a natural mapping $N_G A \times N_G B \to N_G(A \otimes B)$ induced by the tensor product.

Now, to define the cup product for arbitary dimensions, first recall that we can identify $A \otimes B$ with $B \otimes A$ and $A \otimes (B \otimes C)$ with $(A \otimes B) \otimes C$ for G-modules A, B and C. We thus have a natural identifications for the dimension shifting modules $A^p \otimes B = (A \otimes B)^p$ and $A \otimes B^q = (A \otimes B)^q$ for all $p, q \in \mathbb{Z}$. Then, given arbitrary $p, q \in \mathbb{Z}$, we consider the diagram

$$\begin{array}{ccccc}
H^0(G, A^p) \times H^0(G, B^q) & \xrightarrow{\smile} & H^0(G, A^p \otimes B^q) \\
\downarrow \delta^p \quad\quad \downarrow \mathrm{id} & & \downarrow \delta^p \\
H^p(G, A) \times H^0(G, B^q) & \overset{\smile}{\dashrightarrow} & H^q(G, A \otimes B^q) \\
\downarrow \mathrm{id} \quad\quad \downarrow \delta_q & & \downarrow \delta_q \\
H^p(G, A) \times H^q(G, B) & \overset{\smile}{\dashrightarrow} & H^{p+q}(G, A \otimes B)
\end{array} \tag{4}$$

We may then define

$$\smile : H^p(G, A) \times H^q(G, B) \to H^{p+q}(G, A \otimes B)$$

to be the natural homomorphism extending the above diagram to a commutative diagram. It is then immediately clear, by construction, that should $\smile$ satisfy Properties 2 and 3 of the Theorem then $\smile$ is unique.

In order to prove that $\smile$ satisfies Properties 2 and 3, we first find explicit descriptions in the case of $(p, 0)$ and $(0, q)$ for $p, q \geq 0$. We claim that

$$\begin{aligned}
\smile : H^p(G, A) \times H^0(G, B) &\to H^p(G, A \otimes B) \\
(\overline{a_p}, \overline{b_0}) &\mapsto \overline{a_p \otimes b_0}
\end{aligned}$$

$$\begin{aligned}
\smile : H^0(G, A) \times H^q(G, B) &\to H^q(G, A \otimes B) \\
(\overline{a_0}, \overline{b_q}) &\mapsto \overline{a_0 \otimes b_q}
\end{aligned}$$

are the explicit descriptions. It is immediately clear that Property 1 is satisfied by this definition so we must verify Properties 2 and 3. Let us verify Property 2. Suppose that we are given exact sequences

$$0 \longrightarrow A \xrightarrow{\phi} A' \xrightarrow{\psi} A'' \longrightarrow 0$$

$$0 \longrightarrow A \otimes B \xrightarrow{\phi} A' \otimes B \xrightarrow{\psi} A'' \otimes B \longrightarrow 0$$

We need to show that the diagram

$$\begin{array}{ccc}
H^p(G, A'') \times H^0(G, B) & \xrightarrow{\smile} & H^p(G, A'' \otimes B) \\
\downarrow \delta_p \quad\quad \downarrow \mathrm{id} & & \downarrow \delta_p \\
H^{p+1}(G, A) \times H^0(G, B) & \xrightarrow{\smile} & H^{p+1}(G, A \otimes B)
\end{array}$$

commutes. To this end, fix $\overline{a''_p} \in H^p(G, A'')$ and $\overline{b_0} \in H^0(G, B)$. Let a'_p be such that $\psi(a'_p) = a''_p$ and a'_{p+1} be such that $\phi(a_{p+1}) = \partial_{p+1}(a'_p)$. Then $\partial_p(\overline{a''_p}) = \overline{a_{p+1}}$. Then

$$\delta_p(\overline{a_{p''}}) \smile \overline{b_0} = \overline{a_{p+1} \otimes b_0}$$

On the other hand, since δ_p is independent of the choice of preimage, we can choose $a'_p \otimes b_0$ to be a preimage of $a''_p \otimes b_0$ under ψ. Then, clearly, $\phi(a_{p+1} \otimes b_0) = \partial_{p+1}(a'_p \otimes b_0)$. This then implies that

$$\delta_p(\overline{a''_p} \smile \overline{b_0}) = \overline{a'_p \otimes b_0} = \delta_p(\overline{a_{p''}}) \smile \overline{b_0}$$

and so the diagram commutes and Property 2 is satisfied. A similar argument shows that this definition also satisfies Property 3. It is then clear that this definition of $\smile$ coincides with the one given in Diagram 4.

To prove the general case, suppose we are given exact sequences as in the statement of the Theorem. Then we get exact sequences

$$0 \longrightarrow A^q \longrightarrow A'^q \longrightarrow A''^q \longrightarrow 0$$

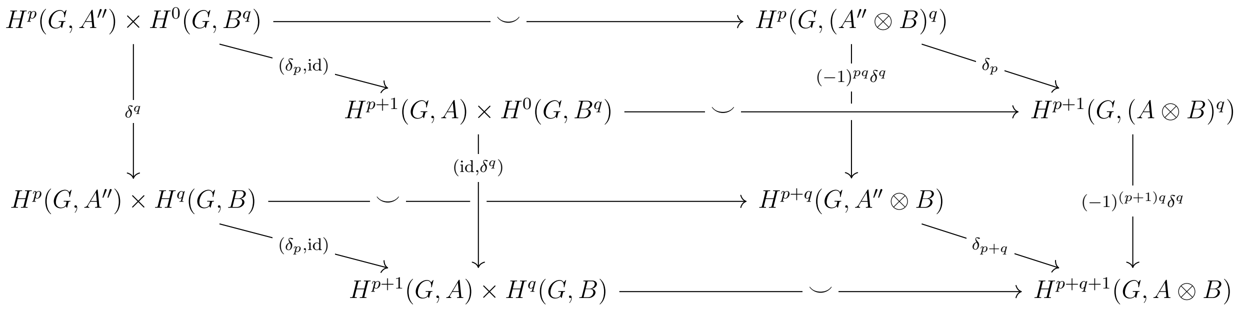
$$0 \longrightarrow (A \otimes B)^q \longrightarrow (A' \otimes B)^q \longrightarrow (A'' \otimes B)^q \longrightarrow 0$$

and

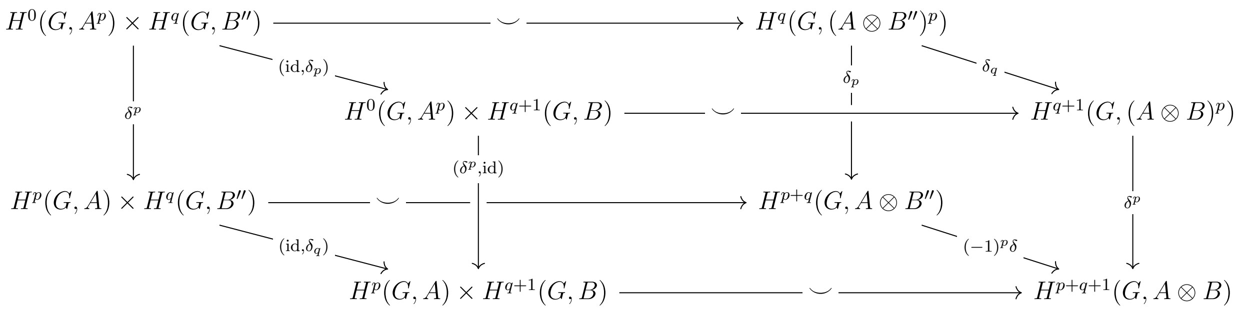
$$0 \longrightarrow B^p \longrightarrow B'^p \longrightarrow B''^p \longrightarrow 0$$

$$0 \longrightarrow (A \otimes B)^p \longrightarrow (A' \otimes B)^p \longrightarrow (A'' \otimes B)^p \longrightarrow 0$$

which induce diagrams



and



Now, the left hand squares of these cubes commutes trivially. The right hand squares commute by the q-fold (respectively p-fold) compositions of squares from Proposition 3.1.6. The front and back squares commute by the definition of the cup product. By the discussion of the cases $(p, 0)$ and $(0, q)$, the top squares commute. Since the vertical maps are all isomorphisms, it then follows that the bottom squares commute. □

5.2 Properties

Proposition 5.2.1. *Let $f : A \to B$ and $g : A' \to B'$ be homomorphisms of G-modules. Denote by $f \otimes g$ the induced homomorphism*

$$f \otimes g : A \otimes B \to A' \otimes B'$$

Then the diagram

$$\begin{array}{ccccc} H^p(G,A) & \times & H^q(G,B) & \xrightarrow{\smile} & H^{p+q}(G, A \otimes B) \\ \downarrow \overline{f} & & \downarrow \overline{g} & & \downarrow \overline{f \otimes g} \\ H^p(G,A') & \times & H^q(G,B') & \xrightarrow{\smile} & H^{p+q}(G, A' \otimes B') \end{array}$$

commutes.

Proof. This is immediate in the case that $p = q = 0$. The general case then follows via dimension shifting. ☐

Proposition 5.2.2. *Let A, B be G-modules and $H \subseteq G$ a subgroup. Then for all $\overline{a} \in H^p(G,A)$ and $\overline{b} \in H^q(G,B)$ we have the relations*

$$\mathrm{res}_p(\overline{a} \smile \overline{b}) = \mathrm{res}_p(\overline{a}) \smile \mathrm{res}_p(\overline{b})$$
$$(\mathrm{cores}_p \circ \mathrm{res}_p)(\overline{a} \smile \overline{b}) = \overline{a} \smile \mathrm{cores}_p(\overline{b})$$

Proof. The general case follows from the case where $p = q = 0$ via dimension shifting. Now suppose that $p = q = 0$. The first formula is immediate. To prove the second formula, fix $a + N_G A \in H^0(G,A)$ and $b + N_H B \in H^0(H,B)$. By the definition of corestriction, we have

$$\begin{aligned} \mathrm{cores}_0((a + N_H A) \cup (b + N_H A)) &= \mathrm{cores}_0(a \otimes b + N_H(A \otimes B) \\ &= \sum_{\sigma \in G/H} ((a \otimes b)^\sigma) + N_G(A \otimes B) \\ &= \left(\sum_{\sigma \in G/H} a \otimes b^\sigma \right) + N_G(A \otimes B) \\ &= \overline{a} \smile \left(\sum_{\sigma G/H} b^\sigma \right) + N_G B \\ &= \overline{a} \smile \mathrm{cores}_0(\overline{b}) \end{aligned}$$

☐

Proposition 5.2.3. *Let A, B and C be G-modules. Suppose that $\overline{a} \in H^p(G,A), \overline{b} \in H^q(G,B)$ and $\overline{s} \in H^r(G,C)$. Then*

1. *The cup product is anti commutative*

$$\overline{a} \smile \overline{b} = (-1)^{pq}(\overline{b} \smile \overline{a})$$

under the canonical isomorphism

$$H^{p+q}(G, A \otimes B) \cong H^{q+p}(G, B \otimes A)$$

2. *The cup product is associatiave*

$$\overline{a} \smile (\overline{b} \smile \overline{c}) = (\overline{a} \smile \overline{b}) \smile \overline{c}$$

under the canonical isomorphism

$$H^{p+q+r}(G, A \otimes (B \otimes C)) \cong H^{p+q+r}(G, (A \otimes B) \otimes C)$$

Proof. The Proposition follows immediately from the properties of the tensor product in dimensions $p = q = r = 0$ and then dimension shfiting for the general cases. ☐

5.3 Explicit Formulae for Low-Dimensional Cup Products

Throughout this section, A and B shall be G-modules. By a_p and b_q, we shall mean a p-cocycle of A and a q-cocycle of B.

Proposition 5.3.1. *We have that* $\overline{a_1} \smile \overline{b_{-1}} = \overline{x_0} \in H^0(G, A \otimes B)$ *where*

$$x_0 = \sum_{\tau \in G} a_1(\tau) \otimes b_{-1}^{\tau}$$

Proof. Recall that we have exact sequences

$$0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$$

$$0 \longrightarrow A \otimes B \longrightarrow A' \otimes B \longrightarrow A'' \otimes B \longrightarrow 0$$

where $A' = \mathbb{Z}[G] \otimes A$ and $A'' = J_G$. We shall identify A with its image in A' and $A \otimes B$ with its image in $A' \otimes B$ in order to ease notation. Since A' is G-induced, it has trivial cohomology. We may thus choose a $a'_0 \in A'$ such that $a_1 = \partial_1(a'_0)$ and

$$a_1(\tau) = a_0'^{\tau} - a'_0$$

for all $\tau \in G$. Let $a''_0 \in A''_G$ be the image of a'_0 in A''. Then $\delta_0(\overline{a''_0}) = \overline{a_1}$. Hence

$$\begin{aligned}
\overline{a_1} \smile \overline{b_{-1}} &= \delta_0(\overline{a''_0}) \smile \overline{b_{-1}} \\
&= \delta_{-1}(\overline{a''_0} \smile \overline{b_{-1}}) && \text{(Theorem 5.1.1)} \\
&= \delta_{-1}(\overline{a''_0 \otimes b_{-1}}) \\
&= \overline{\partial_0(a'_0 \otimes b_{-1})} \\
&= \overline{N_G(a'_0 \otimes b_{-1})} \\
&= \overline{\sum_{\tau \in G} a_0'^{\tau} \otimes b_{-1}^{\tau}} \\
&= \overline{\sum_{\tau \in G} (a_1(\tau) + a'_0) \otimes b_{-1}^{\tau}} \\
&= \overline{\sum_{\tau \in G} [(a_1(\tau) \otimes b_{-1}^{\tau}) + (a'_0 \otimes b_{-1}^{\tau})]} \\
&= \overline{\sum_{\tau \in G} a_1(\tau) \otimes b_{-1}^{\tau}} + \overline{a'_0 \otimes N_G b_{-1}} \\
&= \overline{\sum_{\tau \in G} a_1(\tau) \otimes b_{-1}^{\tau}} && \text{(since } N_G b_{-1} = 0)
\end{aligned}$$

☐

Proposition 5.3.2. *Let $\sigma \in G$ and denote by $\overline{\sigma}$ the element of $H^{-2}(G,\mathbb{Z})$ corresponding to $\sigma[G,G]$ under the isomorphism $H^{-2}(G,\mathbb{Z}) \cong G^{\mathrm{ab}}$. Then*

$$\overline{a_1} \smile \overline{\sigma} = \overline{a_1(\sigma)} \in H^{-1}(G,A)$$

Proof. Recall that we have the exact sequence

$$0 \longrightarrow A \otimes I_G \longrightarrow A \otimes \mathbb{Z}[G] \xrightarrow{\psi} A \longrightarrow 0$$

where ψ is the composite

$$A \otimes \mathbb{Z}[G] \longrightarrow A \otimes \mathbb{Z} \longrightarrow A$$

$$a \otimes \left(\sum_{\sigma\in G} n_\sigma \sigma\right) \longmapsto a \otimes \left(\sum_{\sigma \in G} n_\sigma\right) \longmapsto \sum_{\sigma\in G} n_\sigma a$$

This yields an isomorphism

$$\delta_{-1} : H^{-1}(G,A) \to H^0(G, A \otimes I_G)$$

It thus suffices to show that

$$\delta_{-1}(\overline{a_1} \smile \overline{\sigma}) = \delta_{-1}(\overline{a_1(\sigma)})$$

Choosing the preimage $a_1(\sigma) \otimes 1 = \psi^{-1}(a_1(\sigma))$, and setting $x_0 = \partial_0(a_1(\sigma)\otimes 1)$ we see that

$$\delta_{-1}(\overline{a_1}(\sigma)) = \overline{\partial_0(a_1(\sigma)\otimes 1)} = \overline{N_G(a_1(\sigma)) \otimes 1} = \overline{\sum_{\tau\in G} a_1(\sigma)^\tau \otimes \tau}$$

On the other hand, the isomorphism $\delta_{-2} : H^{-2}(G,\mathbb{Z}) \to H^{-1}(G, I_G)$ sends σ to $\overline{\sigma - 1}$ by Theorem 3.3.6 so we have

$$\begin{aligned} \delta_{-1}(\overline{a_1} \smile \overline{\sigma}) &= -(\overline{a_1} \smile \delta_{-2}(\overline{\sigma})) && \text{(Theorem 5.1.1)} \\ &= -(\overline{a_1} \smile \overline{\sigma - 1}) = \overline{y_0} \end{aligned}$$

Now, Proposition 5.3.1 implies that

$$y_0 = -\left(\sum_{\tau\in G} a_1(\tau)\otimes \tau(\sigma - 1)\right) = \sum_{\tau \in G} a_1(\tau) \otimes \tau - \sum_{\tau\in G} a_1(\tau)\otimes \tau\sigma$$

Since a_1 is a 1-cocycle, we have

$$\begin{aligned} y_0 &= \sum_{\tau\in G} a_1(\tau)\otimes\tau - \sum_{\tau\in G}(a_1(\tau\sigma) - a_1(\sigma)^\tau)) \otimes \tau\sigma \\ &= \sum_{\tau\in G} a_1(\sigma)^\tau \otimes \tau\sigma \end{aligned}$$

Hence

$$y_0 - x_0 = \sum_{\tau\in G} a_1(\sigma)^\tau \otimes \tau(\sigma - 1) = N_G(a_1(\sigma)\otimes(\sigma - 1))$$

Whence $\overline{y_0} = \overline{x_0}$. ☐

Proposition 5.3.3. *Let $\sigma \in G$ and denote by $\overline{\sigma}$ the element of $H^{-2}(G,\mathbb{Z})$ corresponding to $\sigma[G,G]$ under the isomorphism $H^{-2}(G,\mathbb{Z}) \cong G^{\mathrm{ab}}$. Then*

$$\overline{a_2} \smile \overline{\sigma} = \overline{\sum_{\tau \in G} a_2(\tau,\sigma)} \in H^0(G,A)$$

so that we have an induced homomorphism

$$\overline{a_2} \smile - : G^{\mathrm{ab}} \to A^G/N_G A$$

Proof. Consider the exact sequence

$$0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$$

with $A' = A \otimes \mathbb{Z}[G]$ and $A'' = A \otimes J_G$. Since $H^2(G,A') = 0$, there exists a 1-cochain a_1' such that $a_2 = \partial_2(a_1')$ so that

$$a_2(\tau,\sigma) = a_1'(\sigma)^\tau - a_1'(\tau\sigma) + a_1'(\tau)$$

Now, the image a_1'' of a_1' is a 1-cocycle satisfying $\overline{a_2} = \delta_1(a_1'')$. Hence

$$\begin{aligned}
\overline{a_2} \smile \overline{\sigma} &= \delta_1(\overline{a_1''}) \smile \overline{\sigma} \\
&= \delta_0(\overline{a_1''} \smile \overline{\sigma}) && \text{(Theorem 5.1.1)} \\
&= \delta_0(\overline{a_1''(\sigma)}) && \text{(Proposition 5.3.2)} \\
&= \overline{\partial_0(a_1'(\sigma))} \\
&= \overline{\sum_{\tau \in G} a_1'(\sigma)^\tau} \\
&= \overline{\sum_{\tau \in G} a_2(\tau,\sigma) + a_1'(\tau\sigma) - a_1'(\tau)} \\
&= \overline{\sum_{\tau \in G} a_2(\tau,\sigma)}
\end{aligned}$$

□

6 Cohomology of Cyclic Groups

Throughout this section, G will be a cyclic group of order n.

6.1 Cyclic Groups have Periodic Cohomology

Lemma 6.1.1. *Let σ be a generator of G. Then $I_G = \mathbb{Z}[G] \cdot (\sigma - 1)$.*

Proof. Recall that I_G is the free abelian group on $\{\tau - 1\}_{1 \neq \tau \in G}$. Observe that for $k \geq 0$ we have

$$\sigma^k - 1 = (\sigma - 1)(x^{k-1} + x^{k-2} + \cdots + x^1 + 1)$$

and so, in fact, I_G is the principal ideal of $\mathbb{Z}[G]$ generated by $\sigma - 1$. □

Theorem 6.1.2. *Let σ be a generator of G and A a G-module. Then for all $q \in \mathbb{Z}$ we have*

$$H^q(G,A) \cong H^{q+2}(G,A)$$

Proof. It suffices to exhibit an isomorphism $H^{-1}(G, A) \cong H^1(G, A)$. The general case follows via the dimension shifting isomorphisms

$$H^q(G, A) \cong H^{-1}(G, A^{q+1}) \cong H^1(G, A^{q+1}) \cong H^{q+2}(G, A)$$

Now, fix a 1-cocycle a_1 of A. Then for $k \geq 1$ we have

$$\begin{aligned} x(\sigma^k) &= x(\sigma^{k-1})^\sigma + x(\sigma) \\ &= x(\sigma^{k-2})^{\sigma^2} + x(\sigma)^\sigma + x(\sigma) \\ &= \sum_{1 \leq i \leq k-1} x(\sigma)^{\sigma^i} \end{aligned}$$

Hence

$$N_G(x(\sigma)) = \sum_{i=0}^{n-1} x(\sigma)^{\sigma_i} = x(\sigma^n) = x(1) = 0$$

whence $x(\sigma)$ is a (-1)-cocycle of A. Conversely, let $a \in {}_{N_G}A$ be a (-1)-cocycle of A. Then setting

$$x(\sigma^k) = \sum_{i=0}^{k-1} a^{\sigma^i}$$

for all $1 \leq k \leq n-1$ defines a 1-cocycle of A. Hence the assignment $x \mapsto x(\sigma)$ defines an isomorphism $Z_1 \cong Z_{-1}$. Under this isomorphism we have

$$\begin{aligned} x \in R_1 &\iff x(\sigma^k) = a^{\sigma^k} - a && (a \in A) \\ &\iff x(\sigma) = a^\sigma - a && \iff x(\sigma) \in I_G A = R_{-1} \end{aligned}$$

and so 1-coboundaries are mapped to (-1)-coboundaries and we get an induced isomorphism of cohomology groups. □

6.2 Hebrand Quotient

Definition 6.2.1. Let A be an abelian group and $f, g \in \mathrm{End}(A)$ such that $f \circ g = 0$ and $g \circ f = 0$ so that $\mathrm{im}(g) \subseteq \ker(f)$ and $\mathrm{im}(f) \subseteq \ker(g)$. We define the **Herbrand quotient of *A* with respect to *f* and *g*** to be

$$q_{f,g}(A) = \frac{[\ker(f) : \mathrm{im}(g)]}{[\ker(g) : \mathrm{im}(f)]}$$

provided both indices are finite.

Definition 6.2.2. Let A be a G-module and consider the endomorphisms

$$D = \sigma - 1, \quad N = 1 + \sigma + \cdots + \sigma^{n-1}$$

so that $D \circ N = 0 = N \circ D$. Note that

$$\ker(D) = A^G,\ \mathrm{im}(N) = N_G A,\ \ker(N) = {}_{N_G}A,\ \mathrm{im}(D) = I_G A$$

If $H^0(G, A)$ and $H^{-1}(G, A)$ are finite then we say that A is a **Herbrand module** and denote

$$h(A) = q_{D,N}(A) = \frac{[\ker(D) : \mathrm{im}(N)]}{[\ker(N) : \mathrm{im}(D)]} = \frac{|H^0(G, A)|}{|H^{-1}(G, A)|} = \frac{|H^2(G, A)|}{|H^1(G, A)|}$$

Proposition 6.2.3. *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence in $\boldsymbol{G}_{\mathbf{mod}}$*. If any two of* A, B *and* C *are Herbrand modules then so is the third and*

$$h(B) = h(A) \cdot h(C)$$

Proof. Consider the long exact sequence of cohomology groups

$$\begin{array}{ccccc} & H^{-1}(G,A) & \longrightarrow & H^{-1}(G,B) & \\ \nearrow & & & & \searrow \\ H^0(G,C) & & & & H^{-1}(G,C) \\ \nwarrow & & & & \swarrow \\ & H^0(G,B) & \longleftarrow & H^0(G,A) & \end{array}$$

Now recall that given an exact sequence $\{ G_i \}_{1 \leq i \leq n}$ of abelian groups we have the identity $\prod_{i=1}^{n} |G_i|^{(-1)^i} = 1$. It then follows that

$$|H^{-1}(G,A)| \cdot |H^{-1}(G,C)| \cdot |H^0(G,B)| = |H^{-1}(G,B)| \cdot |H^0(G,A)| \cdot |H^0(G,C)|$$

And hence if any two of A, B or C are Herbrand modules, so is the third and

$$h(B) = \frac{|H^0(G,B)|}{H^{-1}(G,B)} = \frac{|H^0(G,A)| \cdot |H^0(G,C)|}{|H^{-1}(G,A)| \cdot |H^{-1}(G,C)|} = h(A)h(C)$$

□

Proposition 6.2.4. *Suppose that* A *is a Herbrand* G*-module with the trivial* G*-action and* $n : A \to A$ *is the multiplication-by-*n *map. Then*

$$h(A) = q_{0,n}(A)$$

Proof. This is immediate from the fact that $D = \sigma - 1$ is identically zero and N_G is just the multiplication-by-n map. □

Corollary 6.2.5. *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence in $\boldsymbol{G}_{\mathbf{mod}}$*. If any two of* $q_{0,n}(A), q_{0,n}(B)$ *and* $q_{0,n}(C)$ *are defined then so is the third and*

$$q_{0,n}(B) = q_{0,n}(A) \cdot q_{0,n}(C)$$

Proposition 6.2.6. *Let* A *be a finite group and* $f, g \in \mathrm{End}(A)$ *such that* $f \circ g = g \circ f = 0$*. Then* $q_{f,g}(A) = 1$*.*

Proof. By an isomorphism theorem, we have that $A/\ker(f) \cong \mathrm{im}(f)$ so that $|A| = |\mathrm{im}(f)| \cdot |\ker(f)|$. On the other hand, we also have that $A/\ker(g) \cong \mathrm{im}(g)$ so that also $|A| = |\mathrm{im}(g)| \cdot |\ker(g)|$. It then follows that $[\ker(f) : \mathrm{im}(g)] = [\ker(g) : \mathrm{im}(f)]$ and so $q_{f,g}(A) = 1$. □

Corollary 6.2.7. *Let A and B be Herbrand G-modules such that A has finite index in B. Then $h(A) = h(B)$.*

Proof. We have that

$$1 = h(A/B) = \frac{h(A)}{h(B)}$$

and so $h(A) = h(B)$ as claimed. □

Lemma 6.2.8. *Let A be an abelian group and $f, g \in \mathrm{End}(A)$. Then*

$$q_{0,gf}(A) = q_{0,g}(A) \cdot q_{0,f}(A)$$

where all three of these quotients are defined when any two of them are.

Proof. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & g(A)\cap\ker(f) & \longrightarrow & g(A) & \xrightarrow{f} & (f\circ g)(A) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \ker(f) & \longrightarrow & A & \longrightarrow & f(A) & \longrightarrow & 0 \end{array}$$

Applying the Snake Lemma yields an exact sequence

$$0 \longrightarrow \ker(f)/(g(A)\cap\ker(f)) \longrightarrow A/g(A) \longrightarrow f(A)/(f\circ g)(A) \longrightarrow 0$$

so that

$$\frac{[A:(g\circ f)(A)]}{[\ker(g\circ f)]} = \frac{[A:g(A)]\cdot|g(A)\cap\ker(f)|}{|\ker(f)|}$$

Now observe that

$$\frac{\ker(f\circ g)}{\ker(f)} = \frac{g^{-1}(g(A)\cap\ker(f))}{g^{-1}(0)} \cong g(A)\cap\ker(f)$$

so that, in fact,

$$\frac{[A:(f\circ g)(A)]}{|\ker(f\circ g)|} = \frac{[A:g(A)]}{|\ker(g)|}\cdot\frac{[A:f(A)]}{|\ker(f)|}$$

Now, this is symmetric in f and g so we get

$$q_{0,gf}(A) = \frac{[A:(g\circ f)(A)]}{|\ker(g\circ f)|} = \frac{[A:g(A)]}{|\ker(g)|}\cdot\frac{[A:f(A)]}{|\ker(f)|} = q_{0,g}(A)\cdot q_{0,f}(A)$$

□

Theorem 6.2.9. *Suppose that G has order prime to p and A a G-module. If $q_{0,p}(A)$ is defined then $q_{0,p}(A)$ is defined and A is a Herbrand module. In particular*

$$h(A)^{p-1} = \frac{q_{0,p}(A^G)^p}{q_{0,p}(A)}$$

Proof. Fix a generator $\sigma \in G$ and let $D = \sigma - 1$. Observe that we have an exact sequence

$$0 \longrightarrow A^G \longrightarrow A \xrightarrow{D} I_G A \longrightarrow 0$$

so that $I_G A$ is a factor group of A. $I_G A$ is also a subgroup of A and so it follows that if $q_{0,p}(A)$ is defined then so is $q_{0,p}(I_G A)$. By Corollary 6.2.5, it follows that $q_{0,n}(A^G)$ is defined and

$$q_{0,p}(A) = q_{0,p}(A^G) \cdot q_{0,p}(I_G A)$$

Moreover, the action of G on A^G is trivial so Proposition 6.2.4 implies that $h(A^G) = q_{0,p}(A^G)$.

We now determine the quotient $q_{0,p}(I_G A)$. By Corollary 1.2.7, the ideal $\mathbb{Z}N_G$ annihilates the $\mathbb{Z}[G]$-module $I_G A$. Hence $I_G A$ has the natural structure of a $\mathbb{Z}[G]/\mathbb{Z}N_G$-module. Now osberve that we have a canonical isomorphism of rings

$$\begin{aligned} \mathbb{Z}[G]/\mathbb{Z}N_G &\to \mathbb{Z}[X]/(1 + X + \cdots + X^{p-1}) \\ \sigma &\to X \end{aligned}$$

By Algebraic Number Theory, the latter ring is isomorphic to the ring of integers $\mathbb{Z}[\zeta]$ of the number field $\mathbb{Q}(\zeta)$ where ζ is a p^{th} root of unity. We thus have an isomorphism

$$\begin{aligned} \mathbb{Z}[G]/\mathbb{Z}N_G &\to \mathbb{Z}[\zeta] \\ \sigma &\to \zeta \end{aligned}$$

By the Elementary Theory of Cyclotomic Fields we have the factorisation

$$p = (\zeta - 1)^{p-1} \cdot e$$

for some $e \in \mathbb{Z}[\zeta]^\times$. We thus have a similar decomposition

$$p = (\sigma - 1)^{p-1} \cdot \varepsilon$$

in $\mathbb{Z}[G]/\mathbb{Z}N_G$ for some unit ε. Now, the endomorphism of $I_G A$ given by multiplication by ε is an automorphism so that $q_{0,\varepsilon}(I_G A) = 1$. Hence

$$\begin{aligned} q_{0,p}(I_G A) &= q_{0,D^{p-1}}(I_G A) \circ q_{0,\varepsilon}(I_G A) && \text{(Lemma 6.2.8)} \\ &= q_{0,D^{p-1}}(I_G A) \\ &= q_{0,D}(I_G A)^{p-1} && \text{(Lemma 6.2.8)} \\ &= q_{D,0}(I_G A)^{1-p} \\ &= q_{D,N}(I_G A)^{1-p} && (N_G \equiv 0) \\ &= h(I_G A)^{1-p} \end{aligned}$$

so we thus have the expressions

$$q_{0,p}(A^G) = h(A^G), \quad q_{0,p}(I_G A) = h(I_G A)^{1-p}, \quad q_{0,p}(A) = h(A^G) \cdot h_{I_G(A)}^{1-p}$$

On the other hand, Proposition 6.2.3 implies that

$$h(A)^{p-1} = h(A^G)^{p-1} \cdot h(I_G A)^{p-1}$$

so that

$$h(A)^{p-1} = \frac{q_{0,p}(A^G)^p}{q_{0,p}(A)}$$

as claimed. ☐

Theorem 6.2.10. *Suppose that G is cyclic of prime order p and let A be a finitely generated G-module. If $\alpha = \mathrm{rank}_{\mathbb{Z}}(A)$ and $\beta = \mathrm{rank}_{\mathbb{Z}}(A^G)$ then*

$$h(A) = p^{(p\beta-\alpha)/(p-1)}$$

Proof. By the Structure Theorem for Finitely Generated Modules over a PID, we have that $A = M \oplus N$ for some torsion module $\mathbb{Z}$ and free $\mathbb{Z}$-module N. Moreover, $\alpha = \mathrm{rank}_{\mathbb{Z}}(A) = \mathrm{rank}_{\mathbb{Z}}(N)$ and $\beta = \mathrm{rank}_{\mathbb{Z}}(A^G) = \mathrm{rank}_{\mathbb{Z}}(N^G)$. Thus Corollary 6.2.7 implies that

$$h(A)^{p-1} = h(N)^{p-1} = \frac{q_{0,p}(N^G)^p}{q_{0,p}(N)}$$

Now,

$$q_{0,p}(N^G) = \frac{[N^G : pN^G]}{|\ker(p)|} = [N^G : pN^G] = p^\beta$$

and

$$a_{0,p}(N) = \frac{[N : pN]}{|\ker(p)|} = p^\alpha$$

so that $h(A) = p^{(p\beta-\alpha)/(p-1)}$ as claimed. □

7 Tate's Theorem

Throughout this section, G shall be a finite group.

Theorem 7.1 (Theorem of Cohomological Triviality)**.** *Let A be a G-module. Suppose there exists $q_0 \in \mathbb{Z}$ such that*

$$H^{q_0}(H, A) = H^{q_0+1}(H, A) = 0$$

for all subgroups $H \subseteq G$. Then A has trivial cohomology.

Proof. It suffices to prove that for all subgroups $H \subseteq G$ the assumption $H^{q_0}(H, A) = H^{q_0+1}(H, A) = 0$ implies that $H^{q_0-1}(H, A) = H^{q_0+2}(H, A) = 0$. Furthermore, it suffices to consider only the case where $q_0 = 1$. The general case follows via dimension shifting.

So assume that $H^1(H, A) = H^2(H, A) = 0$ for all subgroups $H \subseteq G$. We need to show that $H^0(H, A) = H^3(H, A) = 0$ for all subgroups $H \subseteq G$. We shall prove this by induction on $|G|$. Clearly, the case where $|G| = 1$ is trivial. So assume that for all groups C with $1 \leq |C| \leq |G| - 1$, the statement holds. In particular, it holds for all proper subgroups of G so we just need to show that, in fact, the statement holds for G itself. Fix a prime p and suppose that G is not a p-group. Then all the Sylow subgroups of G are necessarily proper subgroups G_l of G and so, by the induction hypothesis, satisfy $H^0(G_l, A) = H^3(G_l, A) = 0$. But Proposition 4.2.7 then implies that $H^0(G, A) = H^3(G, A) = 0$.

We may thus assume that G is a p-group. Let $p^m = |G|$. Sylow's Theorems implies that there exists a subgroup $H \subseteq G$ of order p^{m-1}. Then G/H is cyclic of order p. By the induction hypothesis, we have that

$$H^0(H, A) = H^3(H, A) = 0 \qquad (q = 1, 2, 3)$$

Now, Theorems 4.1.8 and 4.1.9 provide an isomorphism

$$\inf_q : H^q(G/H, A^H) \to H^q(G, A) \qquad (q = 1, 2, 3)$$

Since $H^1(G, A) = 0$ we then have that $H^1(G/H, A^H) = 0$. But G/H is cyclic so applying Theorem 6.1.2 yields

$$0 = H^1(G/H, A^H) \cong H^3(G/H, A^H) \cong H^3(G, A)$$

A similar argument shows that $H^0(G/H, A^H) = 0$. Then

$$\begin{aligned} A^G = (A^H)^{G/H} &\cong N_{G/H} A^H \\ &\cong N_{G/H} N_H A \qquad (H^0(H, A) = 0) \\ &= N_G A \end{aligned}$$

and so $H^0(G, A) = 0$. This completes the proof of the Theorem. □

Theorem 7.2. *Let A be a G-module. Suppose that for each subgroup $H \subseteq G$ we have*

1. $H^{-1}(H, A) = 0$
2. $H^0(H, A)$ *is cyclic of order* $|H|$

If a is a cohomology class generating $H^0(G, A)$ then the map

$$a \smile - : H^q(G, \mathbb{Z}) \to H^q(G, A)$$

is an isomorphism.

Proof. Let $B = A \oplus \mathbb{Z}[G]$ and denote by $i : A \to B$ the canonical injection. Then the induced homomorphism

$$\bar{i} : H^q(H, A) \to H^q(H, B)$$

is an isomorphism for all subgroups $H \subseteq G$. Moreover, $\mathbb{Z}[G]$ has trivial cohomology so it suffices to show that the map $\bar{i} \circ (a \smile -)$ is an isomorphism. To this end, fix a 0-cocycle $a_0 \in A^G$ such that $a = a_0 + N_G A$ is a generator for $H^0(G, A)$. Now consider the map

$$\begin{aligned} f : \mathbb{Z} &\to B \\ n &\mapsto a_0 \cdot n + N_G \cdot n \end{aligned}$$

Then f is clearly injective thanks to the term $N_G \cdot n$. Now let $\overline{c_q} \in H^q(G, \mathbb{Z})$. Then

$$\overline{f}(\overline{c_q}) = \overline{f \circ c_q} = \overline{a_0 \cdot c_q + N_G c_q} = \overline{a_0 \cdot c_q + |G| \cdot c_q} = \overline{a_0 \cdot c_q}$$

since $H^q(G, \mathbb{Z})$ has $|G|$-torsion. On the other hand, we see that

$$a \smile \overline{c_q} = \overline{a_0 \otimes c_q} = \overline{c_q \cdot a_0}$$

via the isomorphism $A \otimes \mathbb{Z} \cong A$ sending $n \otimes a$ to $n \cdot a$. It thus suffices to show that $\overline{f}$ is an isomorphism.

To this end, consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

where C is some G-module and $g : B \to C$ is a map for which $\mathbb{Z}$ is its kernel via f. Now fix a subgroup $H \subseteq G$. By hypothesis we have that

$$0 = H^{-1}(H, A) = H^{-1}(H, B)$$

and $H^1(H, \mathbb{Z})$. Then the long exact cohomology sequence corresponding to the short exact sequence gives

$$0 \longrightarrow H^{-1}(H,\mathbb{Z}) \longrightarrow H^{-1}(H,C) \longrightarrow H^0(H,\mathbb{Z}) \xrightarrow{\overline{f}} H^0(H,B) \longrightarrow H^0(H,C) \longrightarrow 0$$

Since a generates $H^0(G, A) \cong H^0(G, B)$, it follows that $\overline{f}$ is an isomorphism. Hence $H^{-1}(H, C) = H^0(H, C) = 0$. Appealing to Theorem 7.1 shows that, necessarily, $H^q(G, C) = 0$ for all $q \in \mathbb{Z}$. It then follows from the long exact cohomology sequence associated to the short exact sequence above that $\overline{f} : H^q(G, \mathbb{Z}) \to H^q(G, B)$ is an isomorphism as claimed. □

Theorem 7.3 (Tate)**.** *Let A be a G-module. Suppose that for each subgroup $H \subseteq G$ we have*

1. $H^1(H, A) = 0$
2. $H^2(H, A)$ *is cyclic of order* $|H|$

If a is a cohomology class generating $H^2(G, A)$ then the map

$$a \smile - : H^q(G, \mathbb{Z}) \to H^{q+2}(G, A)$$

is an isomorphism. Moreover, for any subgroup $H \subseteq G$, $\mathrm{res}_2(a)$ *generates $H^2(H, A)$ and the map*

$$\mathrm{res}_2(a) \smile - : H^q(H, \mathbb{Z}) \to H^{q+2}(H, A)$$

is an isomorphism.

Proof. Fix a subgroup $H \subseteq G$. Dimension shifting provides us with an isomorphism

$$\delta^2 : H^q(H, A^2) \to H^{q+2}(H, A)$$

so that the assumptions imply that $H^{-1}(H, A^2) = 0$ and $H^0(H, A^2)$ is cyclic of order $|H|$. Moreover, the generator of $H^0(H, A^2)$ is the image of the generator $\delta^{-2}(a) \in H^2(G, A)$. Theorem 7.2 then implies that $\delta^{-2}(a) \smile -$ is an isomorphism. By the definition of the cup product, we have a commutative diagram

$$\begin{array}{ccc} H^q(G,\mathbb{Z}) & \xrightarrow{\delta^{-2}\smile -} & H^q(G,A^2) \\ \downarrow \mathrm{id} & & \downarrow \delta^2 \\ H^q(G,\mathbb{Z}) & \xrightarrow{a\smile -} & H^{q+2}(G,A) \end{array}$$

Since the vertical maps are both isomorphisms, it follows that $a \smile -$ is also an isomorphism.

For the second part of the Theorem, observe that $(\mathrm{cores}_2 \circ \mathrm{res}_2)(a) = [G : H] \cdot a$. Since $H^2(G, A)$ is cyclic of order $|G|$, it follows that $|H|$ divides the order of $\mathrm{res}_2(a)$ and so $\mathrm{res}_2(a)$ generates $H^2(H, A)$. □

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