# The Main Conjecture of Iwasawa Theory over $\mathbb{Q}$

Alexandre Daoud
alex.daoud@cantab.net

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## 1 Cyclotomic Units

**Definition 1.1.** Let $m \neq 2 \pmod 4$[1] and $\zeta_m$ a primitive $m^{th}$ root of unity. We define the group of **cyclotomic units** of $F = \mathbb{Q}(\zeta_m)$ to be the multiplicative group

$$\mathcal{E}_m = \langle \pm\zeta_m, \zeta_m^a - 1 \mid 1 < a < m \rangle \cap \mathcal{O}_F^\times$$

Moreover, we define the **real cyclotomic units** to be $\mathcal{E}_m^+ = \mathcal{E}_m \cap F^+$.

The following is a consequence of the analytic class number formula:

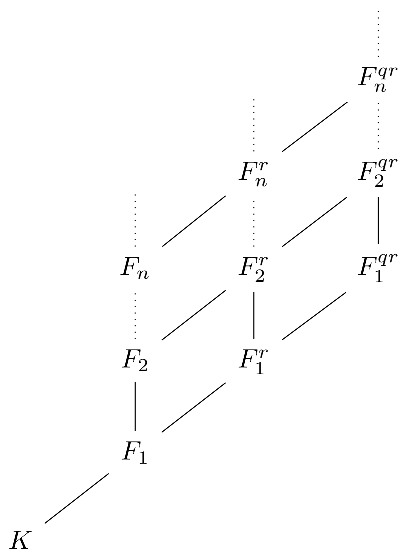
**Theorem 1.2.** *Let $h_m^+$ be the class number of $F = \mathbb{Q}(\zeta_m)^+$. Then*

$$h_m^+ = [\mathcal{O}_{F^+}^\times : \mathcal{E}_m^+] = [\mathcal{O}_F^\times : \mathcal{E}_m]$$

## 2 Euler Systems

For each $k > 1$, fix a primitive $k^{th}$ root of unity $\zeta_k$ such that $\zeta_{kl}^l = \zeta_k$ for all $k$ and $l$.

Fix an odd prime $p$. Let $\mathcal{R}$ be the collection of square-free products of primes coprime to $p$. For each $n \geq 1$, let $F_n = \mathbb{Q}(\zeta_{p^n})^+ = \mathbb{Q}(\zeta_p + \zeta_p^{-1})$. For each $r \in \mathcal{R}$, let $F_n^r = F_n(\zeta_r)$. By $q \in \mathcal{R}$ we shall always mean a prime. Visually, we have the following situation for each $r, q \in \mathcal{R}$:




### 2.1 The Universal Euler System

Denote $G_r = \mathrm{Gal}(F_n^r/F_n)$. Then we have a natural isomorphism $G_r = \prod_{q|r} G_q$.

[1] so that $m$ is the conductor of $\mathbb{Q}(\zeta_m)$

**Definition 2.1.** We define the **norm operator** of $\mathbb{Z}[G_r]$ to be

$$N_r = \prod_{q|r} N_q = \prod_{q|r} \sum_{\sigma_q \in G_q} \sigma_q$$

Now let $\sigma_q$ be a generator of $G_q$. We define the **derivative operator** of $\mathbb{Z}[G_r]$ to be

$$D_r = \prod_{q|r} D_q = \prod_{q|r} \sum_{i=i}^{q-2} i\sigma_q^i$$

**Definition 2.2.** Given $n \in \mathbb{N}$ and $r \in \mathcal{R}$, let $x_{n,r}$ be a symbol. Let $Y_{n,r}$ be the free $\mathbb{Z}[\mathrm{Gal}(F_n^r/F)]$-module on the set $\{ x_{n,s} : s \mid r \}$. Moreover, let $Z_{n,r}$ be the submodule of $Y_{n,r}$ generated by the relations

- $G_{t/s}$ acts trivially on $x_{n,s}$
- If $qs \mid r$ then $N_q x_{n,qs} = (1 - \mathrm{Fr}_q^{-1}) x_{n,s}$

where $\mathrm{Fr}_q$ is the arithmetic Frobenius at $q$ in $\mathrm{Gal}(F_n^s/K)$. Finally, we set $X_{n,r}$ to be the factor module $X_{n,r} = Y_{n,r}/Z_{n,r}$.

**Definition 2.3.** We define the **universal Euler system** to be the direct limit

$$\mathcal{X} = \varinjlim_{n,r} X_{n,r}$$

taken with respect to the norm operators. An **Euler system** is a $G_K$-equivariant map

$$\boldsymbol{\eta} : \mathcal{X} \to \bigcup_{n,r} F_n^{r\times}$$

**Remark.** Specifiying an Euler system is equivalent to specifying a collection of global units

$$\{ \boldsymbol{\eta}(n,r) \in F_n^r \mid n > 1, r \in \mathcal{R} \}$$

satisfying the **norm-compatibility** relations

1. $\mathrm{N}_{F_n^{qr}/F_n^r}\, \boldsymbol{\eta}(n, qr) = \boldsymbol{\eta}(n,r)^{1-\mathrm{Fr}_q^{-1}}$
2. $\mathrm{N}_{F_{n+1}^r/F_n^r}\, \boldsymbol{\eta}(n+1, r) = \boldsymbol{\eta}(n,r)$

**Theorem 2.4.** *For each $n \geq 1$ and $r \in \mathcal{R}$, write $\tau_{n,r} = \mathrm{Fr}_p^{-n}(\zeta_r)$. Define*

$$\boldsymbol{\eta}(n,r) = (\zeta_{p^n}\tau_{n,r} - 1)(\zeta_{p^n}^{-1}\tau_{n,r} - 1)$$

*Then each $\boldsymbol{\eta}(n,r)$ is a cylotomic unit and $\boldsymbol{\eta}$ is an Euler system.*

Here we have used $\tau_{n,r}$ to ensure the second norm-compatibility relation. Without it we can still prove the theorems in the next section but it is nice to have in generality.

## 2.2 Kolyvagin's Derivative Construction

Let $M$ be a power of $p$ and define

$$\mathcal{R}_{n,M} = \{ r \in \mathcal{R} : \forall\, q \mid r, q \text{ splits completely in } F_n \text{ and } q - 1 \equiv 0 \pmod{M} \}$$

**Proposition 2.5.** *Let $r \in \mathcal{R}_{n,M}$. Then $D_r(x_{n,r}) \in (X_{n,r}/MX_{n,r})^{G_r}$.*

**Proposition 2.6.** *Let $\boldsymbol{\eta}$ be an Euler system. Then there exists a $\beta_r \in F_n^{r\times}$ which is unique modulo $F_n^\times$ such that*

$$\frac{\boldsymbol{\eta}(n,r)^{D_r}}{\beta_r^M} \in F_n^{r\times}$$

*We then define a map*

$$\begin{aligned} \kappa_{n,M} : \mathcal{R}_{n,M} &\to F_n^{r\times}/(F_n^{r\times})^M \\ r &\mapsto \left[ \frac{\boldsymbol{\eta}(n,r)^{D_r}}{\beta_r^M} \right] \end{aligned}$$

For the rest of this section, fix $n \in \mathbb{N}$ and denote $L = F_n$. Let $M_L$ be the collection of finite primes of $L$ and $I_L$ the group of fractional ideals of $L$ written additively:

$$I_L = \bigoplus_{\mathfrak{q} \in M_L} \mathbb{Z}\mathfrak{q}$$

Given a finite prime $q$ of $K$, let $I_L^q$ be

$$I_L^q = \bigoplus_{\mathfrak{q}/q} \mathbb{Z}\mathfrak{q}$$

Given $y \in L^\times$, let $(y) \in I_L$ be the principal ideal generated by $y$, and $[y]_q$ the projection of $(y)$ into $I_L^q/MI_L^q$.

**Proposition 2.7.** *Let $q \in \mathcal{R}_{n,M}$. Then there exists a $\mathrm{Gal}(L/K)$-equivariant homomorphism*

$$\phi_q : L^\times/(L^\times)^M \to I_L^q/MI_L^q$$

**Theorem 2.8.** *Let $\boldsymbol{\eta}$ be an Euler system and $q \in \mathcal{R}_{n,M}$. Then*

$$[\kappa_{n,M}(r)]_q = \begin{cases} \phi_q(\kappa_{n,M}(r/q)) & \text{if } q \mid r \\ 0 & \text{if } q \nmid r \end{cases}$$

The following proposition gives us a supply of primes in $\mathcal{R}_{n,M}$ to work with. Let $p > 2$ be prime and $C$ be the $p$-part of the ideal class group of $F = \mathbb{Q}(\zeta_p)^+$.

**Proposition 2.9.** *Let $[\mathfrak{c}] \in C$ be an ideal class, $W$ a finite $G$-submodule of $L^\times/(L^\times)^M$ and a $G = \mathrm{Gal}(L/K)$-equivariant homomorphism*

$$\psi : W \to (\mathbb{Z}/M\mathbb{Z})[G]$$

*Then there are infinitely many primes $\mathfrak{q}$ of $L$ such that*

1. *$q \in \mathcal{R}_{n,M}$ where $q$ is the rational prime lying under $\mathfrak{q}$*
2. *$\mathfrak{q} \in [\mathfrak{c}]$*
3. *For all $w \in W$, $[w]_q = 0$ and there exists $u \in \mathbb{Z}/M\mathbb{Z}^\times$ such that $\phi_q(w) = u\psi(w)\mathfrak{q}$*

# 3 The Main Conjecture

Fix a rational prime $p > 2$. Denote $K_\infty = \bigcup_{n \geq 1} K_n$

$$\Delta = \mathrm{Gal}(K_1/\mathbb{Q}) = (\mathbb{Z}/p\mathbb{Z})^\times, \quad \Gamma = \mathrm{Gal}(K_\infty/K_1) = \mathbb{Z}_p$$

so that $\mathrm{Gal}(K_\infty/\mathbb{Q}) = \Delta \times \mathrm{Gal}(K_\infty/K_1)$. Let $C_n$ be the $p$-part of the ideal class group of $K_n$, $U_n$ the group of principal $p$-units of $K_n$ and $E_n$ the group of global units of $K_n$. Denote

$$\overline{E_n} = \overline{E_n \cap U_n}, \qquad V_n = \overline{\mathcal{E}_n \cap U_n}$$

and

$$C_\infty = \varprojlim_n C_n, \quad E_\infty = \varprojlim_n \overline{E_n}, \quad V_\infty = \varprojlim_n V_n, \quad U_\infty = \varprojlim_n U_n$$

all with respect to norm maps. For $n \leq \infty$, let $\Omega_n$ be the maximal abelian $p$-extension of $K_n$ unramified outside of $p$. Denote $X_n = \mathrm{Gal}(\Omega_n/K_n)$. Let

$$\Lambda = \mathbb{Z}_p[[\Gamma]] = \varprojlim_n \mathbb{Z}_p[\mathrm{Gal}(K_n/K_1)]$$

be the Iwasawa algebra. For each character $\chi \in \widehat{\Delta}$ define the $\chi$-idempotent

$$e_\chi = \frac{1}{p-1} \sum_{\delta \in \Delta} \chi^{-1}(\delta)\delta$$

Given a $\mathbb{Z}_p[\Delta]$-module $Y$, let $Y^\chi = e_\chi Y$ be its $\chi$-isotypical part.

### 3.1 A first consequence of Kolyvagin's Theory

**Theorem 3.1.** *For every character $\chi$ of $\Delta$ and every $n$, $|C_n^\chi|$ divides $|(E_n/\mathcal{E}_n)^\chi|$.*

**Corollary 3.2** (Mazur-Wiles, Kolyvagin)**.** *For every character $\chi$ of $\Delta$ and every $n$, we have*

$$|C_n^\chi| = |(E_n/\mathcal{E}_n)^\chi|$$

*Proof.* By Theorem 1.2 (the analytic class number formula), we have that

$$\prod_\chi |C_n^\chi| = |C_n| = |E_n/\mathcal{E}_n \otimes_{\mathbb{Z}} \mathbb{Z}_p| = \prod_\chi |E_n^\chi/\mathcal{E}_n^\chi|$$

The Corollary then follows by application of the previous Theorem. □

### 3.2 The Main Conjecture

**Theorem 3.3.** *$C_\infty^\chi, E_\infty^\chi, V_\infty^\chi, U_\infty^\chi, X_\infty^\chi$ are all finitely-generated $\Lambda$-modules. $C_\infty^\chi$ is a torsion $\Lambda$-module. If $\chi$ is an even character then $X_\infty^\chi$ and $U_\infty^\chi/V_\infty^\chi$ have $\Lambda$-torsion too.*

Given a finitely generated torsion $\Lambda$-module $M$, there exists a pseudo-isomorphism $M \sim \bigoplus_i \Lambda/f_i\Lambda$. Denote $\mathrm{char}(M) = \prod_i f_i\Lambda$. The Main Conjecture of Iwasawa Theory is the following Theorem of Mazur-Wiles.

**Theorem 3.4** (Mazur-Wiles, Main Conjecture)**.** *For every even character $\chi$ of $\Delta$ we have*

$$(f_\chi) = \mathrm{char}(C_\infty^\chi) = \mathrm{char}((E_\infty/V_\infty)^\chi) = (h_\chi)$$

### 3.3 The Strategy

Let $\gamma$ be a topological generator of $\Gamma$. For each $n \in \mathbb{N}$, let $\Gamma_n = \Gamma/\Gamma^{p^n} = \mathrm{Gal}(K_n/K_1)$. Recall that we have an isomorphism

$$\begin{aligned} \mathbb{Z}_p[\Gamma_n] &\to \mathbb{Z}_p[T]/((1+T)^{p^n} - 1) \\ \gamma &\mapsto 1 + T \end{aligned}$$

Hence letting $I_n = (\gamma^{p^n} - 1)\Lambda$ we have

$$\Lambda_n := \Lambda/I_n \cong \mathbb{Z}_p[\Gamma_n]$$

If $Y$ is a $\Lambda$-module, write

$$Y_{\Gamma_n} = Y/I_nY = Y \otimes_\Lambda \Lambda_n$$

The strategy will be to show that $(f_\chi) = \mathrm{char}(C_\infty^\chi)$ divides $(h_\chi) = \mathrm{char}((E_\infty/V_\infty)^\chi)$. The Main Conjecture will then follow from the following two algebraic lemmas:

**Lemma 3.5.** *Let $\chi$ be an even character of $\Delta$. Then*

1. *For all $n$, $\Lambda_n/f_\chi\Lambda_n$ and $\Lambda_n/h_\chi\Lambda_n$ are finite.*
2. *There is a positive constant $c$ such that for all $n$ we have*

$$c^{-1} \le \frac{|C_n^\chi|}{|\Lambda_n/f_\chi\Lambda_n|} \le c, \qquad c^{-1} \le \frac{|\overline{E_n}^\chi/V_n^\chi|}{\Lambda_n/h_\chi\Lambda_n} \le c$$

**Lemma 3.6.** *Let $a_n \sim b_n$ mean that $a_n/b_n$ is bounded above and below independently of $n$. Let $g_1, g_2 \in \Lambda$ such that $g_1 \mid g_2$ and $|(\Lambda/g_1\Lambda)_{\Gamma_n}| \sim |(\Lambda/g_2\Lambda)_{\Gamma_n}|$. Then $g_1\Lambda = g_2\Lambda$.*

We can now prove the Main Conjecture:

*Proof.* Denote $f = \prod_{\chi \text{ even}} f_\chi$ and $h = \prod_{\chi \text{ even}} h_\chi$. Then the first Lemma and the Mazur-Wiles Theorem imply that

$$
\begin{aligned}
|(\Lambda/f\Lambda)_{\Gamma_n}| \sim \prod_{\chi \text{ even}} |(\Lambda/f_\chi\Lambda)_{\Gamma_n}| \sim \prod_{\chi \text{ even}} |C_n^\chi| = |C_n| = [\overline{E_n}^\chi : V_n^\chi] = \prod_{\chi \text{ even}} |\overline{E_n}^\chi : V_n^\chi| \\
\sim \prod_{\chi \text{ even}} |\Lambda/h_\chi\Lambda| \\
\sim |(\Lambda/h\Lambda)_{\Gamma_n}|
\end{aligned}
$$

By hypothesis, $f \mid h$ so the second Lemma implies that $f\Lambda = g\Lambda$. The division assumption then yields the result. ☐

Hence it suffices to show that $(f_\chi)$ divides $(h_\chi)$.

## 3.4 Some Results from Iwasawa Theory

**Theorem 3.7.** *For every character of $\Delta$, the natural map $(C_\infty^\chi)_{\Gamma_n} \to C_n^\chi$ is an isomorphism. If $\chi$ is even and non-trivial then the natural maps*

$$(X_\infty^\chi)_{\Gamma_n} \to X_n^\chi, \qquad (U_\infty^\chi)_{\Gamma_n} \to U_n^\chi, \qquad (V_\infty^\chi)_{\Gamma_n} \to V_n^\chi$$

*are isomorphisms.*

**Theorem 3.8.** *Let $\chi$ be a non-trivial even character of $\Delta$. Then there is an ideal $\mathcal{A}$ of finite index in $\Lambda$ such that for all $\eta \in \mathcal{A}$ and $n$ there exists a homomorphism $\phi_{n,\eta} : \overline{E_n}^\chi \to \Lambda_n$ such that $\theta_{n,\eta}(V_n^\chi) = \eta h_\chi \Lambda_n$.*

**Theorem 3.9.** *There exists an ideal $\mathcal{B}$ of finite index in $\Lambda$ and for each $n$ ideal classes $\mathfrak{c}_1, \ldots, \mathfrak{c}_n \in C_n^\chi$ such that the annihilator $\mathrm{Ann}(\mathfrak{c}_i)$ of $\mathfrak{c}_i$ in $C_n^\chi/(\Lambda\mathfrak{c}_1 \oplus \cdots \oplus \Lambda\mathfrak{c}_{i-1})$ satisfies $\mathcal{B}\mathrm{Ann}(\mathfrak{c}_i) \subseteq f_i\Lambda_n$ where $f_i$ is the $i^{th}$ "summand" of $f_\chi$*[2]*.*

**Lemma 3.10.** *Let $\chi$ be an even character of $\Delta$. If $\chi$ is trivial then $f_\chi$ and $h_\chi$ are units in $\Lambda$.*

## 3.5 The Proof of the First Division

For this section, we fix $n$ and write $C = C_n, E = \overline{E_n}, V = V_n$ and $F = K_n^+$. Note that if $\chi$ is even then we can identify $C^\chi$ with the $\chi$-part of the $p$-part of the ideal class group of $F$.

Given a power of $p$, $M$, a prime $q \in \mathcal{R}_{n,M}$ and $w \in F^\times$, we write $(w)_q \in I_q$ to be the portion of $(w)$ supported on primes lying above $q$ and $[w]_q$ for its image in $I_q/MI_q$. If $\mathfrak{q}$ is a prime of $F$ lying above $q$ then $I_q^\chi$ is a free $\Lambda_n$-module of rank 1, generated by $\mathfrak{q}^\chi$. Define a map

$$v_\mathfrak{q} = v_{\mathfrak{q},\chi} : F^\times \to \Lambda_n$$

by setting $v_\mathfrak{q}(w)\mathfrak{q}^\chi = (w)_q^\chi$. Write $\overline{v_\mathfrak{q}}$ for the induced map

$$\overline{v_\mathfrak{q}} : F^\times/(F^\times)^M \to \Lambda_n/M\Lambda_n$$

which satisfies $v_\mathfrak{q}(w)\mathfrak{q}^\chi = [w]_q^\chi$.

**Lemma 3.11.** *Fix $r \in \mathcal{R}_{n,M}$, a prime $q \mid r$ and a prime $\mathfrak{q}$ of $F$ lying above $q$. Let $B$ be the subgroup of $C$ generated by the primes of $F$ dividing $r/l$. Let $\mathfrak{c} \in C^\chi$ be the class of $\mathfrak{q}^\chi$ and $W$ the $\Lambda_n$-submodule of $F^\times/(F^\times)^M$ generated by $\kappa_{n,M}(r)^\chi$. If*

1. *$\eta, f \in \Lambda_n$ are such that $\mathrm{Ann}(\mathfrak{c})$ in $\Lambda_n$ of $\mathfrak{c}$ in $C^\chi/B^\chi$ satisfies $\eta\mathrm{Ann}(\mathfrak{c}) \subseteq f\Lambda_n$*
2. *$\Lambda_n/f\Lambda_n$*
3. *$M \geq |C^\chi| \cdot \left|\frac{I_q^\chi/MI_q^\chi}{\Lambda_n[\kappa_{n,M}(r)^\chi]_q}\right|$*

*then there is a Galois-equivariant map $\psi : W \to \Lambda_n/M\Lambda_n$ such that*

$$f\psi(\kappa_{n,M}(r)^\chi) = \eta\overline{v_\mathfrak{q}}(\kappa_{n,M}(r))$$

[2] $C_\infty^\chi$ is pseudoisomorphic to a $\Lambda$-module of the form $\oplus_{i=1}^k \Lambda/(f_i)\Lambda$ so that $f_\chi = \prod_{i=1}^k f_i$

**Theorem 3.12.** *Let $\chi$ be an even character of $\Delta$. Then* $\mathrm{char}(C_\infty^\chi)$ *divides* $\mathrm{char}(E_\infty^\chi / V_\infty^\chi)$.

*Proof.* First suppose that $\chi$ is trivial. Then Lemma 3.10 implies that the characteristic ideals are trivial so the Theorem then follows immediately.

Now suppose that $\chi$ is not trivial. Observe that $\kappa_{n,M}(1)$ is represented by $\xi = \boldsymbol{\eta}(n,1) = (\zeta_{p^n} - 1)(\zeta_{p^n}^{-1} - 1)$ and that $\xi^\chi$ generates $V_n^\chi$. Fix ideal classes $\mathfrak{c}_1, \ldots, \mathfrak{c}_k \in C^\chi$ satisfying Theorem 3.9 3.9. Fix, furthermore, any ideal class $\mathfrak{c}_{k+1} \in C^\chi$. Fix an ideal $\mathcal{C}$ satisfying Theorem 3.8 and Theorem 3.9 (this is possible since the ideals satisfying these Theorems are just annihilators of finite $\Lambda$-modules). Fix $\eta \in \mathcal{C}$ such that $\Lambda_m/\eta\Lambda_m$ is finite for all $m$. Let $\theta := \theta_{n,\eta} : \overline{E_n}^\chi \to \Lambda_n$ be the map provided by Theorem 3.8. Without loss of generality, we may normalise $\theta$ so that $\theta(\xi^\chi) = \eta h_\chi$.

Now let $h$ be any integer such that $p^h \geq |\Lambda_n/\eta\Lambda_n|$ and $p^h \geq |\Lambda_n/h_\chi\Lambda_n|$ which is finite by Lemma 3.10. Set $M = p^{n+(k+1)h}|C^\chi|$.

Using Proposition 2.9 we can inductively choose primes $\mathfrak{q}_1$ of $F$ lying over primes $q_i$ of $\mathbb{Q}$ for $1 \leq i \leq k+1$ such that

$$\lambda_i \in \mathfrak{c}_i, \quad q_i \equiv 1 \pmod{M} \tag{1}$$

$$\overline{v_{\mathfrak{q}_1}}(\kappa_{n,M}(q_1)) = u_1\eta h_\chi, \quad f_{i-1}\overline{v_{\mathfrak{q}_i}}(\kappa_{n,M}(r_i)) = u_i\eta\overline{v_{\mathfrak{q}_{i-1}}}(\kappa_{n,M}(r_{i-1})) \tag{2}$$

where $r_i = \prod_{j \leq i} q_j$ and $u_i \in (\mathbb{Z}/M\mathbb{Z})^\times$.

We only show the basis case: let $\mathfrak{c} = \mathfrak{c}_1$, $W = (E/E^M)^\chi$ and

$$\psi : W \to (\overline{E}/\overline{E}^M)^\chi \xrightarrow{\theta} \Lambda_n/M\Lambda_n \xrightarrow{\chi} (\Lambda_n/M\Lambda_n)^\chi$$

By Proposition 2.9, there exists a prime $\mathfrak{q}_1$ of $F$, a prime $q_i$ of $\mathbb{Q}$ lying below $\mathfrak{q}_1$ and $u_1 \in (\mathbb{Z}/M\mathbb{Z})^\times$ satisfying (1) and such that for all $w \in W, [w]_{q_1} = 0$ and $\phi_{q_1}(w) = u\psi(w)\mathfrak{q}_1$. By the Factorisation Theorem and Proposition 2.9, we have

$$\begin{aligned}\overline{v_{\mathfrak{q}_1}}(\kappa_{n,M}(q_1))\mathfrak{q}_1^\chi = [\kappa_{n,M}(q_1)]_{q_1}^\chi = \phi_{q_1}(\kappa_{n,M}(q_1))^\chi &= u_1\psi(\kappa_{n,M}(q_1))\mathfrak{q}_1^\chi \\ &= u_1\theta(\kappa_{n,M}(q_1)) \\ &= u_1\eta h_\chi \mathfrak{q}_1^\chi\end{aligned}$$

Since $I_{q_1}^\chi / M I_{q_1}^\chi$ is free of rank one over $\Lambda_n/M\Lambda_n$ generated by $\mathfrak{q}_1^\chi$, this proves the basis case.

We now continue this inductive process for $k+1$ steps. Combining all of the relations in (2), we have

$$\eta^{k+1} h_\chi = u f_\chi \overline{v_{\mathfrak{q}_{k+1}}}(\kappa_{n,M}(r_{k+1}))$$

in $\Lambda_n/M\Lambda_n$ for some unit $u \in (\mathbb{Z}/M\mathbb{Z})^\times$. Hence $f_\chi$ divides $\eta^{k+1}h_\chi$ in $\Lambda_n/p^n\Lambda_n$. Since this holds for all $n$, we have that $f_\chi$ divides $\eta^{k+1}h_\chi$ in $\Lambda$.

To remove the factor of $\eta^{k+1}$, note that we can always choose $\eta$ and $\eta'$ relatively prime so that $f_\chi$ divides $\eta^{k+1}h_\chi$ and also ${\eta'}^{k+1}h_\chi$ (for example, let $\eta = p$, $\eta' = \gamma^{p^n} - p$). Since $\Lambda$ is a unique factorisation domain, we necessarily have that $f_\chi \mid h_\chi$. □