Homological Algebra

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Contents

1 Preliminaries

1.1 Note on (Local) Smallness

Throughout this document, we shall assume the Axiom of Universes. That is to say, every set is contained in some large enough Grothendieck universe U. Hence issues of (local) smallness will not be a concern. Indeed the Axiom of Universes ensures that any category $\mathcal{C}$ which is not (locally) small in the regular sense can be made (locally) U-small by fixing a large enough Grothendieck universe U.

1.2 Note on Mitchell's Embedding Theorem

Let $\mathcal{A}$ be a small abelian category. By Mitchell's Embedding Theorem, there exists a ring with unity R such that $\mathcal{A}$ is a full subcategory of Mod_R. We will freely make use of this fact to simplify diagram chasing arguments about abelian categories in the following way. Suppose we are given a diagram $D : J \to \mathcal{A}$ in $\mathcal{A}$ of shape J. Let $\mathcal{A}'$ be the full subcategory of $\mathcal{A}$ that is stable under biproducts and (co)kernels and that contains the image of D. Then $\mathcal{A}'$ is a small abelian category and we can find a ring R such that $\mathcal{A}'$ is a full subcategory of Mod_R. We may then proceed to perform element-wise diagram chases on D in Mod_R and then pullback the results into $\mathcal{A}' \hookrightarrow \mathcal{A}$.

1.3 Filtered Colimits

Definition 1.3.1. Let J be a category. We say that J is **filtered** if the following hold:

1. J is non-empty.
2. Given objects $j, j' \in \operatorname{ob} J$, there exists an object $k \in \operatorname{ob} J$ and morphisms $f : j \to k$, $f' : j' \to k$.
3. Given a parallel pair $u, v : i \to j$ in J, there exists an object $k \in \operatorname{ob} J$ and a morphism $w : j \to k$ such that $wu = wv$.

Definition 1.3.2. Let J be a filtered category and $\mathcal{C}$ a category. Given a diagram $F : J \to \mathcal{C}$ of shape J, we define its **filtered colimit** to be its colimit in $\mathcal{C}$, provided it exists.

Theorem 1.3.3. *Let $\mathbb{T}$ be an algebraic theory and $\mathsf{Alg}_{\mathbb{T}}$ the category of models of $\mathbb{T}$ in Set[1]. Then $\mathsf{Alg}_{\mathbb{T}}$ is complete and cocomplete and the forgetful functor $U : \mathsf{Alg}_{\mathbb{T}} \to \mathsf{Set}$ creates all limits and colimits. In particular, $\mathsf{Alg}_{\mathbb{T}}$ has all filtered colimits which are inherited from Set.*

Proof. Probaly found somwhere in [Bor94]. □

Proposition 1.3.4. *Let $F : J \to \mathsf{Set}$ be a diagram of shape J where J is a filtered category. Then*

$$\varinjlim_{j \in J} Fj = \coprod_{j \in J} Fj / \sim$$

where $\sim$ is the equivalence relation defined as follows: given $x \in Fj, x' \in Fj'$ then $x \sim x'$ if and only if there exists an object $j'' \in \operatorname{ob} J$ and morphisms $(j \xrightarrow{f} j''), (j' \xrightarrow{g} j'') \in \operatorname{mor} J$ such that $Ff(x) = Fg(x)$.

Proof. See [Bor94, Proposition 2.13.3]. □

[1] For example, $\mathsf{Grp}, \mathsf{AbGrp}, \mathsf{Ring}, \mathsf{Mod}_R$...

2 Triangulated Categories

2.1 Definition

Definition 2.1.1. Let $\mathcal{C}$ be an additive category. We say that $\mathcal{C}$ is **triangulated** if there exists the following data:

1. An additive automorphism $T : \mathcal{C} \to \mathcal{C}$ called a **shift** functor. We will often write $A[n] := T^n(A)$ for all $n \in \mathbb{Z}$ and $A \in \operatorname{ob}\mathcal{C}$.

2. A collection of diagrams

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C \xrightarrow{\rho} TA$$

   called **triangles** which we will often write as sextuples $(A, B, C, \phi, \psi, \rho)$ and display as 'diagrams'

$$\begin{array}{ccccc} & & C & & \\ & {\scriptstyle\rho}\swarrow & & \nwarrow{\scriptstyle\psi} & \\ A & & \xrightarrow[\phi]{\qquad} & & B \end{array}$$

   We define a **morphism** of triangles to be a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C & \xrightarrow{\rho} & TA \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Tf} \\ A' & \xrightarrow{\phi'} & B & \xrightarrow{\psi'} & C & \xrightarrow{\rho'} & TA' \end{array}$$

   which we will often abbreviate as a triple (f, g, h).

subject to the following axioms:

<u>TR1</u> Every sextuple $(A, B, C, \phi, \psi, \rho)$ which is isomorphic to a triangle is itself a triangle. Any morphism $A \overset{\phi}{\to} B$ in $\mathcal{C}$ can be embedded in a triangle $(A, B, C, \phi, \psi, \rho)$. The sextuple $(A, A, 0, \mathrm{id}_A, 0, 0)$ is a triangle.

<u>TR2</u> $(A, B, C, \phi, \psi, \rho)$ is a triangle if and only if $(B, C, TA, \psi, \rho, -T\phi)$ is a triangle.

<u>TR3</u> Given triangles $(A, B, C, \phi, \psi, \rho), (A', B', C', \phi', \psi', \rho')$ and morphisms $A \overset{f}{\to} A'$, $B \overset{g}{\to} B'$ such that $g \circ \phi = \phi' \circ f$, there exists a morphism $C \overset{h}{\to} C'$ such that (f, g, h) is a morphism of triangles.

<u>TR4</u> (Octahedral axiom) Suppose we are given three triangles $(L, M, P, \alpha, \gamma, \sigma), (M, N, R, \beta, \varepsilon, \tau)$ and $(L, N, Q, \beta \circ \alpha, \delta, \pi)$. Then there exists a triangle $(P, Q, R, \phi, \psi, \rho)$ making the diagram

$$
\begin{array}{ccccccc}
L & \xrightarrow{\alpha} & M & \xrightarrow{\gamma} & P & \xrightarrow{\sigma} & TL \\
\downarrow{\scriptstyle \mathrm{id}_L} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id}_L} \\
L & \longrightarrow & N & \longrightarrow & Q & \longrightarrow & TL \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}_N} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle T\alpha} \\
M & \longrightarrow & N & \longrightarrow & R & \longrightarrow & TM \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \mathrm{id}_R} & & \downarrow{\scriptstyle T\gamma} \\
P & \xrightarrow{\phi} & Q & \xrightarrow{\psi} & R & \xrightarrow{\rho} & TP
\end{array}
$$

commute.

Remark. The octahedral doesn't play a part throughout this document. It's inclusion in the axioms of a triangulated category is necessary for the theory of t-structures. We call a category $\mathcal{C}$ **pre-triangulated** if it satisfies only axioms TR1 through TR3. For completeness, however, we include the octahedral axiom in the definition but we will not prove that the triangulated categories of interest satisfy it.

Definition 2.1.2. Let $F : \mathcal{C} \to \mathcal{C}$ be an additive functor of triangulated categories. We say that F is a (covariant) **δ-functor** if it commutes with the shift functors and sends triangles to triangles. Dually, a contravariant δ-functor is one that maps triangles to triangles with arrows reversed and the shift functor to its inverse.

Definition 2.1.3. Let $\mathcal{C}$ be triangulated and $\mathcal{A}$ abelian. We say that a functor $H : \mathcal{C} \to \mathcal{A}$ is a **(covariant) cohomological functor** if given a triangle $(A, B, C, \phi, \psi, \rho)$ we have a long exact sequence

$$
\cdots \longrightarrow H(T^iA) \longrightarrow H(T^iB) \longrightarrow H(T^iC) \longrightarrow H(T^{i+1}A) \longrightarrow H(T^{i+1}B) \longrightarrow H(T^{i+1}C) \longrightarrow \cdots
$$

We will often just write $H^i(A) := H(T^iA)$. Dually, we define a contravariant cohomological functor similarly with the arrows reversed.

2.2 Basic Properties

Proposition 2.2.1. *Let $\mathcal{C}$ be a triangulated category.*

1. *The composition of any two consecutive morphisms in a triangle in $\mathcal{C}$ is the zero map.*
2. *If $M \in \operatorname{ob}\mathcal{C}$ then $\mathcal{C}(M, -)$ and $\mathcal{C}(-, M)$ are cohomological functors.*
3. *In the situation of the axiom TR3, if f and g are isomorphisms then so is h.*

Proof. Fix a triangle $\Delta = (A, B, C, \phi, \psi, \rho)$ in $\mathcal{C}$.

Part 1: By TR2, $\Delta' = (B, C, TA, \psi, \rho, -T\phi)$ is also a triangle. It thus suffices to show that $\psi \circ \phi = 0$. By TR1, $\Delta'' = (C, C, 0, \mathrm{id}_C, 0, 0)$ is a triangle. We can now apply TR3 to the triangles Δ' and Δ'' with the morphisms ϕ and id_C to produce a morphism $h : TA \to 0$ giving a morphism of triangles $(\phi, \mathrm{id}_C, h) : \Delta' \to \Delta''$. It then follows that $(T\psi) \circ (-T\phi) = 0$. But T is an automorphism and so $\psi \circ \phi = 0$.

Part 2: Fix $M \in \operatorname{ob}\mathcal{C}$. By TR2, it suffices to show that the sequence

$$
\mathcal{C}(M, A) \xrightarrow{\phi^*} \mathcal{C}(M, B) \xrightarrow{\psi^*} \mathcal{C}(M, C)
$$

is exact. By Part 1, this composition is the zero map so fix a morphism $g \in \mathcal{C}$ such that $\phi^*(g) = 0$. Define $\Delta'' = (M, M, 0, \mathrm{id}_M, 0, 0)$. Applying TR3 to the triangles Δ'' and Δ yields a morphism of triangles $(f, g, h) : \Delta'' \to \Delta$. In particular, the morphism h is such that $g = \phi \circ h = \phi^*(h)$ and so the sequence is exact at $\mathcal{C}(M, B)$.

A similar proof shows that $\mathcal{C}(-, M)$ is a contravariant cohomological functor.

<u>Part 3:</u> Define $\Delta' = (A', B', C', \phi', \psi', \rho')$ and suppose that we have a morphism of triangles $(f, g, h) : \Delta \to \Delta'$ such that f and g are isomorphisms. Applying the cohomological functor $\mathcal{C}(C', -)$, we obtain an exact commuative diagram of abelian groups

$$\begin{array}{ccccccccc} \mathcal{C}(C',A) & \xrightarrow{\phi^*} & \mathcal{C}(C',B) & \xrightarrow{\psi^*} & \mathcal{C}(C',C) & \xrightarrow{\rho^*} & \mathcal{C}(C',TA) & \xrightarrow{(T\phi)^*} & \mathcal{C}(C',TB) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle (Tf)^*} & & \downarrow{\scriptstyle (Tg)^*} \\ \mathcal{C}(C',A') & \xrightarrow{\phi'^*} & \mathcal{C}(C',B') & \xrightarrow{\psi'^*} & \mathcal{C}(C',C') & \xrightarrow{\rho'^*} & \mathcal{C}(C',TA') & \xrightarrow{(T\phi')^*} & \mathcal{C}(C',TB') \end{array}$$

Since f and g are isomorphisms in $\mathcal{C}$, it follows that $f^*, g^*, (Tf)^*$ and $(Tg)^*$ are isomorphisms in AbGrp. Appealing to the Five Lemma then implies that h^* is also an isomorphism. This implies that there exists $y \in \mathcal{C}(C', C)$ such that $h \circ y = \mathrm{id}_{C'}$. An analogous argument with $\mathcal{C}(-, C')$ produces a left-inverse to h so that h is necessarily an isomorphism. □

3 Localisation

3.1 Definition

Definition 3.1.1. Let $\mathcal{C}$ be a category and $S \subseteq \operatorname{mor}\mathcal{C}$ a collection of morphisms. We say that S is a **multiplicative system** if the following axioms hold:

<u>MS1</u> Given $f, g \in S$ such that $f \circ g$ exists then $f \circ g \in S$. For all $A \in \operatorname{ob}\mathcal{C}$, $\mathrm{id}_A \in S$.

<u>MS2</u> Any diagram

$$\begin{array}{ccc} & & Z \\ & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{u} & Y \end{array}$$

with $s \in S$ admits a (not necessarily universal) pullback cone

$$\begin{array}{ccc} W & \xrightarrow{v} & Z \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{u} & Y \end{array}$$

such that $t \in S$. The dual axiom should also hold.

<u>MS3</u> Given a parallel pair $f, g : X \rightrightarrows Y$ in $\mathcal{C}$, the following conditions are equivalent:

i) There exists $s : Y \to Y'$ in S such that $s \circ f = s \circ g$

ii) There exists $t : X' \to X$ in S such that $f \circ t = g \circ t$

Definition 3.1.2. Let $\mathcal{C}$ be a category and $S \subseteq \operatorname{mor}\mathcal{C}$ a collection of morphisms. The **localisation of $\mathcal{C}$ with respect to S** is a category $S^{-1}\mathcal{C}$ (should it exist) equipped with a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ such that

1. For all $s \in S$, $Q(s)$ is an isomorphism.
2. Q is universal amongst functors $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for all $s \in S$.

3.2 The Existence of the Localisation

Lemma 3.2.1. *Let $\mathcal{C}$ be a category and S a multiplicative system. Let S_X be the full subcategory of the slice category $\mathcal{C}/X$ whose objects are morphisms in S. Then S_X is a cofiltered category.*

Proof. S_X is clearly non-empty considering it contains the identity morphism 1_X. Now suppose that we are given objects $(X_1 \xrightarrow{s_1} X), (X_2 \xrightarrow{s_2} X) \in \operatorname{ob} S_X$. By MS2, we have a commutative diagram

$$\begin{array}{ccc} X_3 & \xrightarrow{f} & X_1 \\ \downarrow{\scriptstyle s_3} & & \downarrow{\scriptstyle s_1} \\ X_2 & \xrightarrow{s_2} & X \end{array}$$

for some morphism $s_3 \in S$. By MS1, $s_2 \circ s_3 \in S$ and so $(X_3 \xrightarrow{s_2 \circ s_3} X) \in \operatorname{ob} S_X$. Moreover, this commutative diagram provides us with a morphism $f : (s_3 : X_3 \to X) \to (s_1 : X_1 \to X)$ in the form of the commutative diagram

$$\begin{array}{ccc} X_3 & \xrightarrow{\qquad f \qquad} & X_1 \\ & {\scriptstyle s_2 \circ s_3} \searrow \qquad \swarrow {\scriptstyle s_1} & \\ & X & \end{array}$$

We also trivially have a morphism $(s_3 : X_3 \to X) \to (s_3 : X_2 \to X)$ given by the commutative diagram

$$\begin{array}{ccc} X_3 & \xrightarrow{\qquad s_3 \qquad} & X_2 \\ & {\scriptstyle s_2 \circ s_3} \searrow \qquad \swarrow {\scriptstyle s_2} & \\ & X & \end{array}$$

so that the second axiom of a cofiltered category is satisfied.

Finally, suppose that we are given a parallel pair $f, g : (s_1 : X_1 \to X) \to (s_2 : X_2 \to X)$ in S_X. Then, clearly, $s_2 \circ f = s_2 \circ g$. Then by MS2, there exists an object $t : X_3 \to X$ in S_X such that $f \circ t = g \circ t$. This is precisely the third axiom of a cofiltered category. □

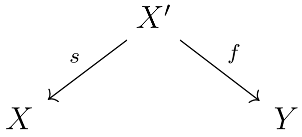
Theorem 3.2.2. *Let $\mathcal{C}$ be a category and S a multiplicative system of morphisms in $\mathcal{C}$. Then the localisation exists and is given explicitly by $S^{-1} \operatorname{ob} \mathcal{C} = \operatorname{ob} \mathcal{C}$ and for all $X, Y \in \operatorname{ob} S^{-1}\mathcal{C}$,*

$$S^{-1}\mathcal{C}(X, Y) = \varinjlim_{X' \in S_X^{\mathrm{op}}} \mathcal{C}(X', Y)$$

where S_X is the full subcategory of the slice category $\mathcal{C}/X$ whose objects are morphisms in S. Moreover, if $\mathcal{C}$ is additive then so is $S^{-1}\mathcal{C}$

Proof. First suppose that $\mathcal{C}$ is additive. Then Theorem 1.3.3 and Proposition 1.3.4 imply that $S^{-1}\mathcal{C}$ is itself enriched over AbGrp.

Proposition 1.3.4 allows us to explicitly describe morphisms in $S^{-1}\mathcal{C}$. Indeed, a morphism $X \to Y$ in $S^{-1}\mathcal{C}$ is an equivalence class in $\coprod_{X' \in S_X^{\mathrm{op}}} \mathcal{C}(X', Y)/\sim$ that is represented by a roof



for some $s \in S$ which we denote as $s^{-1}f$. Proposition 9.1.2 ensures that composition of morphisms in $S^{-1}\mathcal{C}$ is well-defined.

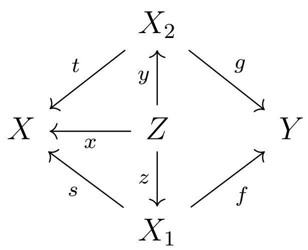
The localisation functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ is the natural one taking a morphism $f : X \to Y$ to the roof $1_X^{-1}f$. Now fix a morphism $s : X \to Y$ in S. It is clear that Qs is an isomorphism with inverse given by the roof $s^{-1}\mathrm{id}_X$.

Finally, we must show that Q is universal amongst functors mapping S to isomorphisms. To this end, fix a functor $T : \mathcal{C} \to \mathcal{D}$ and a morphism $s \in S$ such that Ts is an isomorphism. We need to exhibit a unique functor $G : S^{-1}\mathcal{C} \to \mathcal{D}$ such that $T = G \circ Q$. First suppose that such a G exists. Fix a morphism $\phi \in \operatorname{mor} S^{-1}\mathcal{C}$ represented by a roof $s^{-1}f$. Then, in $S^{-1}\mathcal{C}$, we have that $\phi \circ \mathrm{id}^{-1}s = \mathrm{id}^{-1}f$ so that $\phi \circ Qs = Qf$. Applying G and using the equality $T = G \circ Q$ we have that $G\phi \circ Ts = Tf$. Since Ts is invertible, it follows that $G\phi = Tf \circ (Ts)^{-1}$. Hence T uniquely determines G. We can then use this formula for G on morphisms and equal to TX on objects to obtain a candidate functor G.

Clearly, $G(\mathrm{id}_X) = \mathrm{id}_{TX}$. Now suppose we are given two morphisms $\phi, \psi \in \operatorname{mor} S^{-1}\mathcal{C}$ represented by roofs $s^{-1}f$ and $t^{-1}g$ respectively. Then $\psi \circ \phi$ is represented by a roof $(s \circ u)^{-1}(g \circ v)$ where $u \in S$ and v are morphisms such that $tv = fu$. This implies that $Tt \circ Tv = Tf \circ Tu$ so that $Tv \circ (Tu)^{-1} = (Tt)^{-1}) \circ Tf$. Then

$$\begin{aligned} G(\psi \circ \phi) = T(g \circ v) \circ (T(s \circ u))^{-1} &= Tg \circ Tv \circ (Tu)^{-1} \circ (Ts)^{-1} \\ &= Tg \circ (Tt)^{-1} \circ Tf \circ (Ts)^{-1} \\ &= G\psi \circ G\phi \end{aligned}$$

We must now show that G is independent of the choice of representative of the morphism $\phi : X \to Y \in \operatorname{mor} S^{-1}\mathcal{C}$. Suppose that ϕ is represented by two roofs $s^{-1}f : X \xrightarrow{s^{-1}} X_1 \xrightarrow{f} Y$ and $t^{-1}g : X \xrightarrow{t^{-1}} X_2 \xrightarrow{g} Y$. Then there exist an object Z and morphisms $x : Z \to X$ in $S, y : Z \to X_1$ and $z : Z \to X_2$ such that the diagram



commutes. Then $Tf \circ Tz = Tg \circ Ty$ and $Ts \circ Tz = Tt \circ Ty$. Observe that since $sz = x \in S$, $Ts \circ Tz = T(s \circ z)$ is invertible. But $s \in S$ as well so that Ts is invertible whence so is Tz. Similarly, Ty is an isomorphism. Hence $(Tz)^{-1} \circ (Ts)^{-1} = (Ty)^{-1} \circ (Tt)^{-1}$. Then

$$\begin{aligned} Tf \circ (Ts)^{-1} = Tf \circ Tz \circ (Tz)^{-1} \circ (Ts)^{-1} &= Tf \circ Tz \circ (Ty)^{-1} \circ (Tt)^{-1} \\ &= Tg \circ Ty \circ (Ty)^{-1} \circ (Tt)^{-1} \\ &= Tg \circ (Tt)^{-1} \end{aligned}$$

so that $G\phi$ is independent of the choice of representative roof of ϕ. □

Remark. We could have dually shown that the coslice category ${}_Y S$ is filtered and defined morphisms in $S^{-1}\mathcal{C}$ as

$$S^{-1}\mathcal{C}(X,Y) = \varinjlim_{Y' \in {}_Y S} \mathcal{C}(X,Y')$$

This is precisely the purpose of the presence of the dual statements in axioms MS2 and MS3.

3.3 Examples and Properties of the Localisation

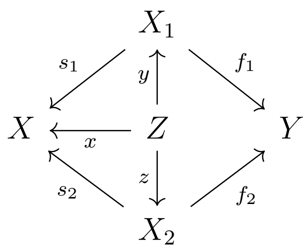
Example 3.3.1. Let R be a commutative ring with unity and S a multiplicative set in R. We can view R as a category with one object whose morphisms are the elements of R and composition given by the multiplication. Then the categorical localisation $S^{-1}R$ coincides with the well-known ring-theoretic localisation.

Proposition 3.3.2. *Let $\mathcal{C}$ be a category and S a multiplicative system in $\mathcal{C}$. Given a full subcategory $\mathcal{D}$ of $\mathcal{C}$, suppose that $S' = S \cap \bigcup_{X,Y} \mathcal{D}(X,Y)$ is a multiplicative system for $\mathcal{D}$. Assume that one of the following conditions holds:*

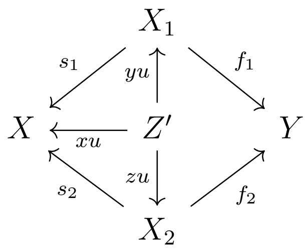
1. *If $x : Z \to X$ is a morphism in S such that $X \in \operatorname{ob}\mathcal{D}$ then there is a morphism $u : Z' \to Z$ with $Z' \in \operatorname{ob} D$ and such that $xu \in S'$.*
2. *The dual of* (1).

Then the natural functor $F : S'^{-1}\mathcal{D} \to S^{-1}\mathcal{C}$ is fully faithful.

Proof. We first show that F is faithful. To this end, suppose that $s_1^{-1} f_1 : X \to X_1 \to Y$ and $f_2^{-1} f_2 : X \to X_2 \to Y$ are two roofs in $S'^{-1}\mathcal{D}$ and assume that their image under F coincides. Then in $\mathcal{C}$ we have an object Z together with morphisms $x : Z \to X$ in $S, y : Z \to X_1$ and $z : Z \to X_2$ such that the diagram



commutes. By hypothesis, there exists an object $Z' \in \mathcal{D}$ together with a morphism $u : Z' \to Z$ such that $xu \in S$. Since $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ it follows that x, y and z are all morphisms in $\mathcal{D}$. Moreover, $xu \in S'$. We then have a commutative diagram



in $\mathcal{D}$. But this is exactly what it means for the two roofs $s_1^{-1} f_1$ and $s_2^{-1} f_2$ to be equivalent. Hence F is faithful.

The fact that F is full is an immediate consequence of the fact that $\mathcal{D}$ is itself a full subcategory of $\mathcal{C}$. $\square$

Proposition 3.3.3. *Let $\mathcal{C}$ be a category, S a multiplicative system for $\mathcal{C}$ and $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ the localisation functor. Given another category $\mathcal{D}$ let $F, G : S^{-1}\mathcal{C} \to \mathcal{D}$ be functors. Then there exists a bijection*

$$\Phi : [S^{-1}\mathcal{C}, \mathcal{D}](F, G) \to [\mathcal{C}, \mathcal{D}](F \circ Q, G \circ Q)$$

between natural transformations from F to G and natural transformations from $F \circ Q$ to $G \circ Q$.

Proof. We define Φ to be the natural mapping

$$\begin{aligned} \Phi : [S^{-1}\mathcal{C}, \mathcal{D}](F, G) &\to [\mathcal{C}, \mathcal{D}](F \circ Q, G \circ Q) \\ \alpha &\mapsto \alpha_Q \end{aligned}$$

This map is clearly injective since $\operatorname{ob} S^{-1}\mathcal{C} = \operatorname{ob}\mathcal{C}$. To see that α is surjective, fix a natural transformation $\beta : FQ \to GQ$. Then for all $X \in \operatorname{ob} S^{-1}$, we have a morphism $\beta_X : FQX \to GQX$ in $\mathcal{D}$ such that for every morphism $f : X \to Y$ in $\mathcal{C}$ there exists a commutative diagram

$$\begin{array}{ccc} FQX & \xrightarrow{FQ(f)} & FQY \\ \downarrow{\scriptstyle \beta_X} & & \downarrow{\scriptstyle \beta_Y} \\ GQX & \xrightarrow{GQ(f)} & GQY \end{array}$$

in $\mathcal{D}$. Define a candidate natural transformation $\alpha : F \to G$ by $\alpha_X = \beta_X$ for all $X \in \operatorname{ob}\mathcal{C}$. It is then obvious that $\Phi(\alpha) = \beta$ but we need to verify that α is itself a natural transformation. To this end, let $\phi : X \to Y$ be a morphism in $S^{-1}\mathcal{C}$ represented by a roof $s^{-1}f : X \to X' \to Y$. We need to verify that the following diagram is commutative:

$$\begin{array}{ccc} FX & \xrightarrow{F(\phi)} & FY \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ GX & \xrightarrow{G(\phi)} & GY \end{array}$$

This is equivalent to the commuativity of the diagram

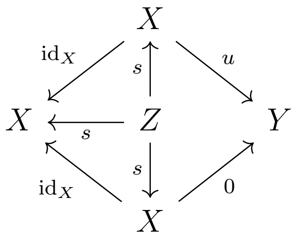
$$\begin{array}{ccccc} FQX & \xrightarrow{(FQs)^{-1}} & FQX' & \xrightarrow{FQ(f)} & FQY \\ \downarrow{\scriptstyle \beta_X} & & \downarrow{\scriptstyle \beta_{X'}} & & \downarrow{\scriptstyle \beta_Y} \\ GQX & \xrightarrow{(GQs)^{-1}} & GQX' & \xrightarrow{GQ(f)} & GQY \end{array}$$

The commutativity of the left hand square is a consequence of the invertibilty of FQs together with the definition of β. The right hand square's commutativity is also a consequence of the definition of β. □

Proposition 3.3.4. *Let $\mathcal{C}$ be an additive category, S a multiplicative system and $u : X \to Y$ a morphism in $\mathcal{C}$. Then the following are equivalent:*

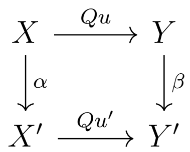
1. $Qu = 0$
2. *There exists $s : Z \to X$ in S such that $us = 0$*
3. *There exists $t : Y \to Z'$ in S such that $tu = 0$*

Proof. The equivalence of (2) and (3) follows immediately from MS3. Fix $s : Z \to X$ in S such that $us = 0$. Then $(Qu) \circ (Qs) = 0$. But Qs is an isomorphism whence $Qu = 0$. Hence (2) implies (1). Now assume that $Qu = 0$. Qu admits a representation by the roof $\mathrm{id}_X^{-1}u$ whence we have an equivalence of roofs $\mathrm{id}_X^{-1}u = \mathrm{id}_X^{-1}0$. Hence there exists a morphism $s : Z \to X$ in S such that we have a commutative diagram

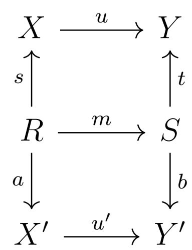


In particular, $s \in S$ is such that $us = 0s = 0$ which is exactly (2). □

Corollary 3.3.5. *Let $\mathcal{C}$ be an additive category and S a multiplicative system in $\mathcal{C}$. Suppose we are given a commutative diagram*



in $S^{-1}\mathcal{C}$. Then there exists a roof $s^{-1}a : X \to R \to X'$ representing α and a roof $t^{-1}b : Y \to S \to Y'$ representing β, together with a morphism $m : R \to S$ in $\mathcal{C}$ such that the diagram



commutes.

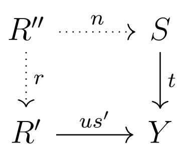
Proof. Fix a representative roof $t^{-1}b : Y \to S \to Y'$ of β and a representative roof $s'^{-1}a'^{-1} : X \to R' \to X'$ of α. Then

$$\alpha = Q(a') \circ Q(s')^{-1}$$
$$\beta = Q(b) \circ Q(t')^{-1}$$

By hypothesis, we then have that

$$Q(b) \circ Q(t)^{-1} \circ Q(u) = Q(u') \circ Q(a') \circ Q(s')^{-1}$$

By MS2, there exist morphisms $r : R'' \to R'$ in S and $n : R'' \to S$ such that the diagram



commutes from which we establish the diagram

$$\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
{\scriptstyle s'r}\uparrow & & \uparrow{\scriptstyle t} \\
R'' & \xrightarrow{n} & S \\
{\scriptstyle a'r}\downarrow & & \downarrow{\scriptstyle b} \\
X' & \xrightarrow{u'} & Y'
\end{array}$$

in which the upper square commutes. To establish the "commutativity" of the lower square observe that the hypothesis together with the commutativity of the upper square imply that

$$\begin{aligned}
Q(u'a'r - bn) &= Q(u') \circ Q(a') \circ Q(r) - Q(b) \circ Q(n) \\
&= Q(b) \circ Q(t)^{-1} \circ Q(u) \circ Q(s') \circ Q(r) - Q(b) \circ Q(n) \\
&= Q(b) \circ Q(t)^{-1} \circ Q(u) \circ Q(s'r) - Q(b) \circ Q(n) \\
&= Q(b) \circ Q(t)^{-1} \circ Q(t) \circ Q(n) - Q(b) \circ Q(n) \\
&= 0
\end{aligned}$$

Hence there exists $r' : R \to R''$ in S such that $u'a'rr' = bnr'$. Setting $s = s'rr' \in S, a = a'rr'$ and $m = nr'$ yields the Corollary. □

Corollary 3.3.6. *Let $\mathcal{C}$ be an additive category and S a multiplicative system in $\mathcal{C}$. If $u : X \to Y$ is a monomorphism (resp. epimorphism) then Qu is a monomorphism (resp. epimorphism).*

Proof. Fix $\alpha : Z \to X$ in $S^{-1}\mathcal{C}$ such that $Q(u) \circ \alpha = 0$. Suppose that α is represented by a roof $s^{-1}f : X \to X' \to Y$. Then

$$0 = Q(u) \circ \alpha = Q(u) \circ Q(f) \circ Q(s)^{-1} = Q(uf) \circ Q(s)^{-1}$$

Hence $Q(uf) = 0$. By Proposition 3.3.4, there exists $t \in S$ such that $uft = 0$ in $\mathcal{C}$. But u is a monomorphism and so $ft = 0$. Proposition 3.3.4 then implies that $Q(f) = 0$. Hence $\alpha = Q(f) \circ Q(s)^{-1} = 0$ whence $Q(u)$ is monic. The statement for epimorphisms follows dually. □

3.4 Transport of Triangulation to the Localisation

Definition 3.4.1. Let $\mathcal{C}$ be a triangulated category and S a multiplicative system in $\mathcal{C}$. We say that S is **compatible with the triangulation** if the following axioms are satisfied:

<u>MS4</u> $s \in S$ if and only if $Ts \in S$.

<u>MS5</u> If (f, g, h) is a morphism of triangles with $f, g \in S$ then $h \in S$.

Theorem 3.4.2. *Let $\mathcal{C}$ be a triangulated category and S a multiplicative system in $\mathcal{C}$ compatible with the triangulation. Then $S^{-1}\mathcal{C}$ inherits a unique triangulation from $\mathcal{C}$ making the localisation functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ into a δ-functor. Furthermore, Q is universal amongst δ-functors $T : \mathcal{C} \to \mathcal{D}$ of triangulated categories that map S to isomorphisms.*

Proof. We define a triangulated structure on $S^{-1}\mathcal{C}$ as follows. By MS4, the composition $Q \circ T$ maps morphisms in S to isomorphisms. By the universal property of Q, there thus exists a unique functor $S^{-1}T : S^{-1}\mathcal{C} \to S^{-1}\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{T} & \mathcal{C} \\ \downarrow{\scriptstyle Q} & & \downarrow{\scriptstyle Q} \\ S^{-1}\mathcal{C} & \xrightarrow{S^{-1}T} & S^{-1}\mathcal{C} \end{array}$$

commutes. It then follows that $S^{-1}T$ is an automorphism which we denote from now on by T by abuse of notation.

Take as triangles in $S^{-1}\mathcal{C}$ all diagrams isomorphic to the images of triangles of $\mathcal{C}$ under the action of the localisation functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$. It is now immediate from the definition that, should these data provide a triangulated structure for $S^{-1}\mathcal{C}$, Q is a δ-functor.

We now verify the axioms of a triangulated category.

<u>TR1:</u> Every sextuple $(A, B, C, \phi, \psi, \rho)$ which is isomorphic to a triangle is a triangle by definition. Now fix a morphism $\phi : X \to Y$ in $S^{-1}\mathcal{C}$. We need to exhibit a triangle $(X, Y, Z, \phi, \psi, \rho)$. Choose a representative roof $s^{-1}f : X \to X' \to Y$ of ϕ. By TR1 for $\mathcal{C}$, we can find a triangle (X', Y, Z, f, g, h). Applying the localisation functor yields the triangle (X', Y, Z, Qf, Qg, Qh) in $S^{-1}\mathcal{C}$. Now consider the commutative diagram

$$\begin{array}{ccccccc} X' & \xrightarrow{Qf} & Y & \xrightarrow{Qg} & Z & \xrightarrow{Qh} & TX' \\ \downarrow{\scriptstyle Qs} & & \downarrow{\scriptstyle 1_Y} & & \downarrow{\scriptstyle 1_Z} & & \downarrow{\scriptstyle T\circ Qs} \\ X & \xrightarrow{\phi} & Y & \xrightarrow{Qg} & Z & \xrightarrow{T\circ Q(s\circ h)} & TX \end{array}$$

in $S^{-1}\mathcal{C}$. The vertical arrows are clearly isomorphisms and so $(X, Y, Z, \phi, Qg, T \circ Q(s \circ c))$ is a triangle in $S^{-1}\mathcal{C}$. Finally, the sextuple $(X, X, 0, \mathrm{id}_X, 0, 0)$ is clearly a triangle in $S^{-1}\mathcal{C}$ since it is the image of the identity triangle in $\mathcal{C}$ under Q.

<u>TR2:</u> Fix a triangle $\Delta = (X, Y, Z, \phi, \psi, \rho)$ in $S^{-1}\mathcal{C}$. We need to show that $\Delta' = (Y, Z, TX, \psi, \rho, -T\phi)$ is also a triangle. By the definition of the candidate triangulation on $S^{-1}\mathcal{C}$, $(X, Y, Z, \phi, \psi, \rho)$ is the isomorphic image under Q of some triangle (X, Y, Z, f, g, h) in $\mathcal{C}$. By TR2 for $\mathcal{C}$, we have that $(Y, Z, TZ, g, h, -Tf)$ is also a triangle. The image under Q of this triangle is then clearly isomorphic to Δ'.

<u>TR3:</u> Fix triangles $(X, Y, Z, \phi, \psi, \rho)$ and $(X', Y', Z', \phi', \psi', \rho')$ in $S^{-1}\mathcal{C}$ and suppose they arise from triangles (A, B, C, u, v, w) and (A', B', C', u', v', w') in $\mathcal{C}$ such that we have the following commutative diagrams in $S^{-1}\mathcal{C}$

$$\begin{array}{ccccccc} X & \xrightarrow{\phi} & Y & \xrightarrow{\psi} & Z & \xrightarrow{\rho} & TX \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle T(\xi)} \\ A & \xrightarrow{Q(u)} & B & \xrightarrow{Q(v)} & C & \xrightarrow{Q(c)} & TA \end{array} \qquad \begin{array}{ccccccc} X' & \xrightarrow{\phi'} & Y' & \xrightarrow{\psi'} & Z' & \xrightarrow{\rho'} & TX' \\ \downarrow{\scriptstyle \xi'} & & \downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \zeta'} & & \downarrow{\scriptstyle T(\xi')} \\ A' & \xrightarrow{Q(u')} & B' & \xrightarrow{Q(v')} & C' & \xrightarrow{Q(c')} & TA' \end{array}$$

with vertical maps isomorphisms. Now suppose that we have a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\phi} & Y & \xrightarrow{\psi} & Z & \xrightarrow{\rho} & TX \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \mu} & & \vdots & & \downarrow{\scriptstyle T(\lambda)} \\ X' & \xrightarrow{\phi'} & Y' & \xrightarrow{\psi'} & Z' & \xrightarrow{\rho'} & TX' \end{array}$$

We need to exhibit a morphism $\nu : Z \to Z'$ in $S^{-1}\mathcal{C}$ such that the triple (λ, μ, ν) is a morphism of triangles. Combining these three commutative diagrams provides us with a commutative diagram in $S^{-1}\mathcal{C}$

$$
\begin{array}{ccccccc}
A & \xrightarrow{Q(u)} & B & \xrightarrow{Q(v)} & C & \xrightarrow{Q(w)} & TA \\
\downarrow{\scriptstyle \xi'\lambda\xi^{-1}} & & \downarrow{\scriptstyle \eta'\mu\eta^{-1}} & & & & \downarrow \\
A' & \xrightarrow{Q(u')} & B' & \xrightarrow{Q(v')} & C' & \xrightarrow{Q(w')} & TA'
\end{array}
$$

By Corollary 3.3.5, we can find a representative roof $s^{-1}a : A \to R \to A'$ of $\xi'\lambda\xi^{-1}$, a represntative roof $t^{-1}b : B \to S \to B'$ of $\eta'\mu\eta^{-1}$ and a morphism $m : R \to S$ in $\mathcal{C}$ fitting in a diagram

$$
\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & TA \\
\uparrow{\scriptstyle s} & & \uparrow{\scriptstyle t} & & {\scriptstyle r}\,\vdots\!\uparrow & & \uparrow{\scriptstyle T(s)} \\
R & \xrightarrow[m]{} & S & \overset{n}{\dashrightarrow} & M & \overset{o}{\dashrightarrow} & TR \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \vdots\!\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle T(a)} \\
A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & TA'
\end{array}
$$

such that the squares consisting of filled in arrows commute. By TR3, we can find a sextuple (R, S, M, m, n, o) making the central row a triangle in $\mathcal{C}$. TR1 then implies that we can find r and c making the triples (s, t, r) and (a, b, c) into morphisms of triangles. Moreover, MS5 implies that $r \in S$. Now let ν' be the equivalence class of the roof $r^{-1}c$ in $S^{-1}\mathcal{C}$. Then the morphism $\nu = \zeta'^{-1}\nu'\zeta$ is the desired morphism making the triple (λ, μ, ν) into a morphism of triangles in $S^{-1}\mathcal{C}$.

TR4: The proof of TR4 follows the same principle as TR3 but we omit its (lengthy) proof.

□

4 The Derived Category

4.1 The Category of Complexes

Definition 4.1.1. Let $\mathcal{C}$ be a category. We define a **graded $\mathcal{C}$-object** to be a family $X^\bullet = \{ X^i \}_{i\in\mathbb{Z}}$ of objects X^i of $\mathcal{C}$. The object X^i is called the **i^{th} component** of $X^\bullet$.

We denote by $\mathcal{C}^p(X^\bullet, Y^\bullet)$ the collection of **graded morphisms of degree** $p \in \mathbb{Z}$. That is to say, the set of families of morphisms $f^\bullet = \{ f^i \}_{i\in\mathbb{Z}}$ such that $f^i \in \mathcal{C}(X^i, Y^{i+p})$. Given graded $\mathcal{C}$-objects $X^\bullet, Y^\bullet, Z^\bullet, f^\bullet \in \mathcal{C}^p(X^\bullet, Y^\bullet)$ and $g \in \mathcal{C}^q(X^\bullet, Y^\bullet)$, we denote by $|f|$ the degree of f and define $fg \in \mathcal{C}^{p+q}(X^\bullet, Z^\bullet)$ by setting $(fg)^i = f^{i+p}g^i$.

Definition 4.1.2. Let $\mathcal{C}$ be an additive category. We define a **complex** in $\mathcal{C}$ to be a pair $(X^\bullet, d_X)$ consisting of a graded $\mathcal{C}$-object toegether with a graded morphism $d_X \in \mathcal{C}^1(X^\bullet, X^\bullet)$ called the **differential** of $X^\bullet$ such that $d_X d_X = 0$. We will often omit the subscript for the differential when it is clear what complex we are discussing. A complex will often be viewed as a sequence

$$
\cdots \longrightarrow X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \longrightarrow \cdots
$$

We define a **morphism of complexes** $f : X^\bullet \to Y^\bullet$ to be a graded morphism $f \in \mathcal{C}^0(X^\bullet, Y^\bullet)$ such that $fd_X = d_Y f$. In other words, we require that the diagram

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & X^{i-1} & \xrightarrow{d_X^{i-1}} & X^i & \xrightarrow{d_X^i} & X^{i+1} & \longrightarrow & \cdots \\
& & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\
\cdots & \longrightarrow & Y^{i-1} & \xrightarrow{d_Y^{i-1}} & Y^i & \xrightarrow{d_Y^i} & Y^{i+1} & \longrightarrow & \cdots
\end{array}
$$

commutes. We denote by $\mathsf{Com}(\mathcal{C})$ the category whose objects are the complexes in $\mathcal{C}$ and whose morphisms are the morphisms of complexes.

Proposition 4.1.3. *Let $\mathcal{A}$ be an additive (resp. abelian) category. Then $\mathsf{Com}(\mathcal{A})$ is also additive (resp. abelian).*

Proof. This follows immediately by taking all the necessary structure (i.e AbGrp-enrichment, (co)limits, (co)images) to be defined component-wise and then using the fact that $\mathcal{A}$ is additive (resp. abelian). □

Remark. Let $\mathcal{A}$ be an additive category.

1. Given a complex $X^\bullet$ in $\mathsf{Com}(\mathcal{A})$, we say that $X^\bullet$ is a **complex concentrated at degree j** if $X^i = 0$ for all $i \neq j$ and $X^j = X$ for some object X of $\mathcal{A}$.

2. There is a canonical fully faithful functor $\mathcal{A}_n : \mathcal{A} \to \mathsf{Com}(\mathcal{A})$ sending objects to the corresponding complex concentrated at degree n together with the trivial differential. Given an object $A \in \mathcal{A}$, we denote $\mathcal{A}_n(A) = A[n]^\bullet$.

Definition 4.1.4. Let $\mathcal{A}$ be an abelian category, $X^\bullet$ a complex in $\mathsf{Com}(\mathcal{A})$ and $q \in \mathbb{Z}$. We define the **q^{th} cocycle object, coboundary object and cohomology object** of $X^\bullet$ to be

$$\begin{aligned} Z^q(X^\bullet) &= \ker d_X^q \\ B^q(X^\bullet) &= \operatorname{im} d_X^{q-1} \\ H^q(X^\bullet) &= \operatorname{coker}(B^q(X^\bullet) \to Z^q(X^\bullet)) \\ &= \operatorname{coker}(\operatorname{im} d_X^{q-1} \to \ker d_X^q) \end{aligned}$$

respectively. This clearly defines an exact functor $H^q : \mathsf{Com}(\mathcal{A}) \to \mathcal{A}$. We say that $X^\bullet$ is **acyclic** if $H^q(X^\bullet) = 0$ for all $q \in \mathbb{Z}$.

Definition 4.1.5. Let $\mathcal{A}$ be an additive category. We define the following full subcategories of $\mathsf{Com}(\mathcal{A})$:

1. $\mathsf{Com}^+(\mathcal{A})$: The category of bounded below complexes in $\mathcal{A}$. In other words, $\mathsf{Com}^+(\mathcal{A})$ consists of all complexes $X^\bullet$ for which $X^i = 0$ for $i \ll 0$.

2. $\mathsf{Com}^-(\mathcal{A})$: The category of bounded above complexes in $\mathcal{A}$. In other words, $\mathsf{Com}^-(\mathcal{A})$ consists of all complexes $X^\bullet$ for which $k \in \mathbb{Z}$ with $X^i = 0$ for $i \gg 0$.

3. $\mathsf{Com}^b(\mathcal{A}) = \mathsf{Com}^+(\mathcal{A}) \cap \mathsf{Com}^-(\mathcal{A})$

4.2 The Homotopy Category

Definition 4.2.1. Let $\mathcal{A}$ be an additive category and $u \in \mathsf{Com}(\mathcal{A})(X^\bullet, Y^\bullet)$ a morphism of complexes. We say that u is **null-homotopic** if there exists a graded morphism $k \in \mathcal{A}^{-1}(X^\bullet, Y^\bullet)$ such that $u = d_Y k + k d_X$. Given another morphism $v \in \mathsf{Com}(\mathcal{A})(X^\bullet, Y^\bullet)$, we say that u is **homotopic** to v, denoted $u \sim v$, if $u - v$ is null-homotopic.

Proposition 4.2.2. *Let $\mathcal{A}$ be an additive category. Then the homotopy relation on $\mathsf{Com}(\mathcal{A})(X^\bullet, Y^\bullet)$ is an equivalence relation compatible with composition of morphisms.*

Proof. Fix a morphism $u \in \mathsf{Com}(\mathcal{A})(X^\bullet, Y^\bullet)$. Then $u - u = 0 \sim 0$ so that u is homotopic to itself. Fix another morphism $v \in \mathsf{Com}(\mathcal{A})(X^\bullet, Y^\bullet)$ and suppose that $u \sim v$. Then $u - v = d_Y k + k d_X$ for some $k \in \mathcal{A}^{-1}(X^\bullet, Y^\bullet)$. Define $k' \in \mathcal{A}^{-1}(X^\bullet, Y^\bullet)$ to be $k' = -k$. Then

$$v - u = -(u - v) = -(d_Y k + k dX) = d_Y k' + k' d_X$$

so that $v \sim u$. Fix a further morphism $w \in \mathsf{Com}(\mathcal{A})(X^\bullet, Y^\bullet)$ and suppose that $u \sim v$ and $v \sim w$. Then $u - v = d_Y k + k d_X$ and $v - w = d_Y m + m d_X$ for some $k, m \in \mathcal{A}^{-1}(X^\bullet, Y^\bullet)$. It then follows that

$$u - w = u - v + v - w = d_Y k + k d_X + d_Y m + m d_X = d_Y(k + m) + (k + m) d_X$$

so that $u \sim w$. To show that $\sim$ is compatible with composition in $\mathcal{A}$, suppose that we are given $f_1, f_2 \in \mathsf{Com}(\mathcal{A})(X^\bullet, Y^\bullet)$ and $g_1, g_2 \in \mathsf{Com}(\mathcal{A})(Y^\bullet, Z^\bullet)$ such that $f_1 \sim f_2$ and $g_1 \sim g_2$. Then $f_1 - f_2 = d_Y k + k d_X$ and $g_1 - g_2 = d_Z l + l d_Z$ for some $k, l \in \mathcal{A}^{-1}(X^\bullet, Y^\bullet)$. We need to show that $g_1 f_1 \sim g_2 f_2$. Observe that

$$\begin{aligned}
g_1 f_1 - g_2 f_2 &= g_1 f_1 - g_2 f_1 + g_2 f_1 - g_2 f_2 \\
&= (g_1 - g_2) f_1 + g_2 (f_1 - f_2) \\
&= (d_Z l + l d_Y) f_1 + g_2 (d_Y k + k d_X) \\
&= d_Z l f_1 + l d_Y f_1 + g_2 d_Y k + g_2 k d_X \\
&= d_Z l f_1 + l f_1 d_X + d_Z g_2 k + g_2 k d_X \\
&= d_Z (l f_1 + g_2 k) + (l f_1 + g_2 k) d_X
\end{aligned}$$

so that $l f_1 + g_2 k$ is a homotopy from $g_1 f_1$ to $g_2 f_2$. □

Definition 4.2.3. Let $\mathcal{A}$ be an additive category and $\sim$ the relation of homotopy equivalence on $\mathsf{Com}A$. We define the **homotopy category** of $\mathcal{A}$ to be the quotient category $\mathsf{K}(\mathcal{A}) = \mathsf{Com}(\mathcal{A})/\sim$. For $* \in \{\varnothing, +, -, b\}$, we define the full subcategory $\mathsf{K}^*(\mathcal{A})$ of $\mathsf{K}(\mathcal{A})$ to be the image of $\mathsf{Com}^*(\mathcal{A})$ under the canonical quotient functor $\mathsf{Com}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$.

Proposition 4.2.4. *Let $\mathcal{A}$ be an additive category. Then $\mathsf{K}(\mathcal{A})$ is additive.*

Proof. This is immediate from the fact that the quotient category of an additive category is additive. □

Remark. Note that it is not the case that if $\mathcal{A}$ is abelian then $\mathsf{K}(\mathcal{A})$ is abelian.

Proposition 4.2.5. *Let $\mathcal{A}$ be an abelian category. Then the cohomology functor H^q : $\mathsf{Com}(\mathcal{A})) \to \mathcal{A}$ descends to a functor*

$$H^q : \mathsf{K}(\mathcal{A}) \to \mathcal{A}$$

Proof. We need to show that H^q is invariant on homotopy classes of morphisms of $\mathsf{Com}(\mathcal{A})$. To this end, suppose that $u, v \in \mathsf{Com}(\mathcal{A})(X^\bullet, Y^\bullet)$ are two homotopic morphisms so that $u - v = d_Y k + k d_X$ for some $k \in \mathcal{A}^{-1}(X^\bullet, Y^\bullet)$. Then the restriction to $\ker(d_X^q)$ of H^q is the restriction to $\ker(d_X^q)$ of $d_Y k + k d_X$. Hence $H^q(u - v) = d_Y k$. Moreover, passing to the cokernel

$$\operatorname{coker}(\operatorname{im} d_Y^{q-1} \to \ker d_Y^q)$$

implies that $H^q(u - v) = 0$. Hence $H^q(u) = H^q(v)$. □

4.3 Triangulating the Homotopy Category

Definition 4.3.1. Let $\mathcal{A}$ be an additive category. We define the **shift functor** $T : \mathsf{Com}^*(\mathcal{A}) \to \mathsf{Com}^*(\mathcal{A})$ to be the additive automorphism that associates to every complex $(X^\bullet, d_X)$ the complex $(T(X^\bullet), d_{T(X^\bullet)})$ where $T(X^\bullet)^i = X^{i+1}$ for all $i \in \mathbb{Z}$ and $d_{T(X^\bullet)} = -d_X$. We will often denote $TX^\bullet$ as just $X[1]^\bullet$.

Lemma 4.3.2. *Let $\mathcal{A}$ be an abelian category. Then the shift functor $T : \mathsf{Com}^*(\mathcal{A}) \to \mathsf{Com}^*(\mathcal{A})$ descends to an additive automorphism*

$$T : \mathsf{K}^*(\mathcal{A}) \to \mathsf{K}^*(\mathcal{A})$$

Proof. We need to check that $T : \mathsf{Com}^*(\mathcal{A}) \to \mathsf{Com}^*(\mathcal{A})$ is constant on homotopy classes of morphisms in $\mathsf{Com}^*(\mathcal{A})(X^\bullet, Y^\bullet)$. To this end, fix $f, g \in \mathsf{Com}^*(\mathcal{A})(X^\bullet, Y^\bullet)$ and suppose that $k \in \mathcal{C}^{-1}(X^\bullet, Y^\bullet)$ is a homotopy from f to g so that $f - g = d_Y k + k d_X$. Then, in $\mathsf{Com}(\mathcal{A})$, we have

$$\begin{aligned} T(f-g)^i = f^{i+1} - g^{i+1} &= d_Y^i k^{i+1} + k^{i+2} d_X^i \\ &= -d_{TY}^i k^{i+1} - k^{i+2} d_{TX}^i \end{aligned}$$

so that $-k$ is a homotopy from Tf to Tg. It then follows that in $\mathsf{K}(\mathcal{A})$ we have $Tf = Tg$. $\square$

Definition 4.3.3. Let $\mathcal{A}$ be an additive category and $f \in \mathsf{Com}^*(\mathcal{A})(X^\bullet, Y^\bullet)$ a morphism. We define the **mapping cone** of f to be the complex given by the data

$$\begin{aligned} C(f)^\bullet &= X[1]^\bullet \oplus Y^\bullet \\ d_f^i &= \begin{pmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{pmatrix} \end{aligned}$$

The mapping cone can be visualised as a diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X^i & \xrightarrow{-d_X^i} & X^{i+1} & \xrightarrow{-d_X^{i+1}} & X^{i+2} & \longrightarrow & \cdots \\ & & \oplus & \searrow^{f^i} & \oplus & \searrow^{f^{i+1}} & \oplus & & \\ \cdots & \longrightarrow & Y^{i-1} & \xrightarrow{-d_Y^{i-1}} & Y^i & \xrightarrow{-d_Y^i} & Y^{i+1} & \longrightarrow & \cdots \end{array}$$

Moreover if $\mathcal{A}$ is abelian the canonical projection $\pi_f : C(f)^\bullet \to X[1]^\bullet$ and inclusion $\nu_f : Y^\bullet \to C(f)^\bullet$ induce a canonical exact sequence

$$0 \longrightarrow Y^\bullet \xrightarrow{\nu_f} C(f)^\bullet \xrightarrow{\pi_f} X[1]^\bullet \longrightarrow 0$$

of complexes in $\mathsf{Com}(\mathcal{A})$.

Lemma 4.3.4. *Let $\mathcal{A}$ be an additive category, $X^\bullet \in \mathsf{Com}^*(\mathcal{A})$ a complex and $C(\mathrm{id}_{X^\bullet})$ the mapping cone of the identity morphism $\mathrm{id}_{X^\bullet}$. Then the identity morphism $\mathrm{id}_{C(\mathrm{id}_{X^\bullet})}$ is null-homotopic. In particular, $C(\mathrm{id}_{X^\bullet})$ is the zero complex in $\mathsf{K}^*(\mathcal{A})$.*

Proof. Consider the morphism of complexes defined by

$$k^i = \begin{pmatrix} 0 & \mathrm{id}_{X^\bullet}^i \\ 0 & 0 \end{pmatrix}$$

We claim that k is a homotopy from $\mathrm{id}_{C(\mathrm{id}_{X^\bullet})}$ to the 0 map. Indeed,

$$\begin{aligned}
d_f^{i-1}k^i + k^{i+1}d_f^i &= \begin{pmatrix} -d_{X^\bullet}^i & 0 \\ \mathrm{id}_{X^\bullet}^i & d_{X^\bullet}^{i-1} \end{pmatrix}\begin{pmatrix} 0 & \mathrm{id}_{X^\bullet}^i \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{id}_{X^\bullet}^{i+1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -d_{X^\bullet}^{i+1} & 0 \\ \mathrm{id}_{X^\bullet}^{i+1} & d_{X^\bullet}^i \end{pmatrix} \\
&= \begin{pmatrix} 0 & -d_{X^\bullet}^i \mathrm{id}_{X^\bullet}^i \\ 0 & \mathrm{id}_{X^\bullet}^i \end{pmatrix} + \begin{pmatrix} \mathrm{id}_{X^\bullet}^{i+1} & \mathrm{id}_{X^\bullet}^{i+1} d_{X^\bullet}^i \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{id}_{X^\bullet}^{i+1} & 0 \\ 0 & \mathrm{id}_{X^\bullet}^i \end{pmatrix} \\
&= \mathrm{id}_{C(\mathrm{id}_{X^\bullet})}
\end{aligned}$$

□

Proposition 4.3.5. *Let $\mathcal{A}$ be an additive category. Suppose that $f, g \in \mathsf{Com}^*(\mathcal{A})(X^\bullet, Y^\bullet)$ are homotopic morphisms. Then $C(f)^\bullet \cong C(g)^\bullet$.*

Proof. Suppose that $k \in \mathcal{A}^{-1}(X^\bullet, Y^\bullet)$ is a homotopy between f and g so that $f - g = d_Y k + k d_X$. We claim that the map

$$u^i = \begin{pmatrix} \mathrm{id}_{X^\bullet}^{i+1} & 0 \\ -k^{i+1} & \mathrm{id}_{Y^\bullet}^i \end{pmatrix}$$

is an isomorphism of mapping cones $u : C(g)^\bullet \to C(f)^\bullet$. We must first check that it is a morphism. Observe that

$$\begin{aligned}
u^{i+1}d_g^i &= \begin{pmatrix} \mathrm{id}_{X^\bullet}^{i+2} & 0 \\ -k^{i+2} & \mathrm{id}_{Y^\bullet}^{i+1} \end{pmatrix}\begin{pmatrix} -d_X^{i+1} & 0 \\ g^{i+1} & d_Y^i \end{pmatrix} \\
&= \begin{pmatrix} -\mathrm{id}_{X^\bullet}^{i+2} d_X^{i+1} & 0 \\ g^{i+1} + k^{i+2}d_X^{i+1} & \mathrm{id}_{Y^\bullet}^{i+1} d_Y^i \end{pmatrix} \\
&= \begin{pmatrix} -d_X^{i+1}\mathrm{id}_{X^\bullet}^{i+2} & 0 \\ f^{i+1} - d_Y^i k^{i+1} & d_Y^i \mathrm{id}_{Y^\bullet}^{i+1} \end{pmatrix} \\
&= \begin{pmatrix} -d_{X^i}^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{pmatrix}\begin{pmatrix} \mathrm{id}_{X^\bullet}^{i+1} & 0 \\ -k^{i+1} & \mathrm{id}_{Y^\bullet}^i \end{pmatrix} \\
&= d_f^i u^i
\end{aligned}$$

It is now easy to see that u is an isomorphism with inverse given by the map

$$v^i = \begin{pmatrix} \mathrm{id}_{X^\bullet}^{i+1} & 0 \\ k^{i+1} & \mathrm{id}_{Y^\bullet}^i \end{pmatrix}$$

□

Corollary 4.3.6. *Let $\mathcal{A}$ be an abelian category and $f \in \mathsf{Com}^*(X^\bullet, Y^\bullet)$ a morphism of complexes. Then the canonical projection $\pi_f : C(f)^\bullet \to X[1]^\bullet$ and inclusion $\nu_f : Y^\bullet \to C(f)^\bullet$ are compatible with homotopy.*

Proof. By the Proposition, we have that $C(f) \cong C(g)$ and so $\pi_f = \pi_g$. A similar argument applies to ν_f and ν_g. □

Definition 4.3.7. Let $\mathcal{A}$ be an abelian category. We define a **standard triangle** in $\mathsf{K}^*(\mathcal{A})$ to be a diagram of the form

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{\nu_f} C(f)^\bullet \xrightarrow{\pi_f} X[1]^\bullet$$

Theorem 4.3.8. *Let $\mathcal{A}$ be an abelian category. Then $\mathsf{K}^*(\mathcal{A})$ is triangulated by the following data:*

1. *The shift functor is given by $T : \mathsf{K}^*(\mathcal{A}) \to \mathsf{K}^*(\mathcal{A})$.*
2. *The triangles are given by the collection of all diagrams in $\mathsf{K}^*(\mathcal{A})$ that are isomorphic to standard triangles.*

Proof.

TR1: The first two parts of TR1 are immediate by the definitions. Lemma 4.3.4 ensures that the sextuple $(X^\bullet, X^\bullet, 0, \mathrm{id}_X, 0, 0)$ is indeed a triangle.

TR2: It suffices to prove TR2 for the standard triangle $(X^\bullet, Y^\bullet, C(f)^\bullet, f, \nu_f, \pi_f)$. We claim that the triangle $(Y^\bullet, C(f)^\bullet, X[1]^\bullet, \nu_f, \pi_f, -f[1])$ is isomorphic to the standard triangle $(Y^\bullet, C(f)^\bullet, C(\nu_f)^\bullet, \nu_f, \nu_{\nu_f}, \pi_{\nu_f})$ for ν_f. To this end, define morphisms

$$\begin{aligned} \phi : X[1]^\bullet &\to C(\nu_f)^\bullet \\ \phi^i &= (-f^{i+1}, \mathrm{id}_{X^\bullet}^{i+1}, 0) \end{aligned}$$

and

$$\begin{aligned} \psi : C(\nu_f)^\bullet &\to X[1]^\bullet \\ \psi^i &= (0, \mathrm{id}_{X^\bullet}^{i+1}, 0) \end{aligned}$$

We claim that ϕ and ψ are mutually inverse and make the diagram

$$\begin{array}{ccccccc} Y^\bullet & \xrightarrow{\nu_f} & C(f)^\bullet & \xrightarrow{\pi_f} & X[1]^\bullet & \xrightarrow{-f[1]} & Y[1]^\bullet \\ \downarrow{\scriptstyle \mathrm{id}_{Y^\bullet}} & & \downarrow{\scriptstyle \mathrm{id}_{C(f)^\bullet}} & & {\scriptstyle \phi}\downarrow\uparrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{id}_{Y[1]^\bullet}} \\ Y^\bullet & \xrightarrow{\nu_f} & C(f)^\bullet & \xrightarrow{\nu_{\nu_f}} & C(\nu_f)^\bullet & \xrightarrow{\pi_{\nu_f}} & Y[1]^\bullet \end{array}$$

commute so that $(\mathrm{id}_{Y^\bullet}, \mathrm{id}_{C(f)^\bullet}, \phi)$ and $(\mathrm{id}_{Y^\bullet}, \mathrm{id}_{C(f)^\bullet}, \psi)$ are isomorphisms of triangles. First observe that $\pi_{\nu_f} \circ \phi = -f[1]$ so that the downwards direction right-hand square commutes. Similarly, $\psi \circ \nu_{\nu_f} = \pi_f$ so that the upwards direction central square commutes.

To see that the downwards direction central square commutes, consider the morphism

$$u = \begin{pmatrix} 0 & -\mathrm{id}_{Y^\bullet} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : (C(f)^\bullet = X[1]^\bullet \oplus Y^\bullet) \to (C(\nu_f)^\bullet = Y[1]^\bullet \oplus X[1]^\bullet \oplus Y^\bullet)$$

Then

$$
\begin{aligned}
d^{i-1}_{\nu_f} u^i + u^{i+1} d^i_f &= \begin{pmatrix} -d^i_Y & 0 & 0 \\ 0 & -d^i_X & 0 \\ \nu^i_f & f^i & d^{i-1}_Y \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{id}^i_{Y^\bullet} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{id}^{i+1}_{Y^\bullet} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -d^{i+1}_X & 0 \\ f^{i+1} & d^i_Y \end{pmatrix} \\
&= \begin{pmatrix} 0 & d^i_Y \mathrm{id}^i_Y \\ 0 & 0 \\ -\mathrm{id}^i_{Y^\bullet} & 0 \end{pmatrix} + \begin{pmatrix} -\mathrm{id}^{i+1}_{Y^\bullet} f^{i+1} & -\mathrm{id}^{i+1}_Y d^i_Y \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} f^{i+1} & 0 \\ 0 & 0 \\ -\mathrm{id}^i_{Y^\bullet} & 0 \end{pmatrix} \\
&= \begin{pmatrix} f^{i+1} & 0 \\ 0 & \mathrm{id}^{i+1}_{X^\bullet} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id}^{i+1}_{X^\bullet} \\ \mathrm{id}^i_{Y^\bullet} & 0 \end{pmatrix} \\
&= \phi \circ \pi_f - \nu_{\nu_f}
\end{aligned}
$$

so that $\phi \circ \pi_f$ is homotopic to ν_{ν_f}. Hence in $\mathsf{K}^*(\mathcal{A})$, we have that $\phi \circ \pi_f = \nu_{\nu_f}$ and the downwards direction central square commutes. A similar argument with the homotopy $v = (0, 0, \mathrm{id}_{Y[1]^\bullet})$ ensures that the upwards direction right-hand square commutes.

It remains to show that ϕ and ψ are mutually inverse. But this follows from the fact that $\psi \circ \phi = \mathrm{id}_{X[1]^\bullet}$ and $\phi \circ \psi \sim \mathrm{id}_{C(\nu_f)}$ via the homotopy map

$$
\begin{pmatrix} 0 & 0 & -\mathrm{id}_{Y^\bullet} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : C(\nu_f)^\bullet \to C(\nu_f)^\bullet
$$

<u>TR3:</u> It suffices to prove TR3 for standard triangles. Suppose we are given a commutative diagram

$$
\begin{array}{ccccccc}
X^\bullet & \xrightarrow{f} & Y^\bullet & \xrightarrow{\nu_f} & C(f)^\bullet & \xrightarrow{\pi_f} & X[1]^\bullet \\
\big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \beta} & & \vdots{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle \alpha[1]} \\
X'^\bullet & \xrightarrow{f'} & Y'^\bullet & \xrightarrow{\nu_{f'}} & C(f')^\bullet & \xrightarrow{\pi_{f'}} & X'[1]^\bullet
\end{array}
$$

in $\mathsf{K}^*(\mathcal{A})$. Then there exists a homotopy u between $f' \circ \alpha$ and $\beta \circ f$. We claim that

$$
\gamma^i = \begin{pmatrix} \alpha^{i+1} & 0 \\ -u^{i+1} & \beta^i \end{pmatrix} : C(f)^\bullet \to C(f')^\bullet
$$

completes the above diagram to a morphism of triangles. We must first verify that γ is indeed a morphism of complexes $\gamma : C(f)^\bullet \to C(f')^\bullet$. Indeed,

$$
\begin{aligned}
d^i_{f'} \gamma^i &= \begin{pmatrix} -d^{i+1}_X & 0 \\ f'^{i+1} & d^i_Y \end{pmatrix} \begin{pmatrix} \alpha^{i+1} & 0 \\ -u^{i+1} & \beta^i \end{pmatrix} \\
&= \begin{pmatrix} -d^{i+1}_X \alpha^{i+1} & 0 \\ f'^{i+1}\alpha^{i+1} - d^i_Y u^{i+1} & d^i_Y \beta^i \end{pmatrix} \\
&= \begin{pmatrix} -\alpha^{i+2} d^{i+1}_X & 0 \\ \beta^{i+1} f^{i+1} + u^{i+2} d^{i+1}_X & \beta^{i+1} d^i_Y \end{pmatrix} \\
&= \begin{pmatrix} \alpha^{i+2} & 0 \\ -u^{i+2} & \beta^{i+1} \end{pmatrix} \begin{pmatrix} -d^{i+1}_X & 0 \\ f^{i+1} & d^i_Y \end{pmatrix} \\
&= \gamma^{i+1} d^i_f
\end{aligned}
$$

It is immediate from the definition of γ that the above diagram is completed to a commutative diagram in $\mathsf{K}^*(\mathcal{A})$.

<u>TR4:</u> Omitted.

$\square$

Proposition 4.3.9. *Let $\mathcal{A}$ be an abelian category. Then the cohomology functor H^0 : $\mathsf{K}^*(\mathcal{A}) \to \mathcal{A}$ is cohomological.*

Proof. It suffices to prove the Proposition for standard triangles. To this end, fix a morphism of complexes $f : X^\bullet \to Y^\bullet$. Then we have a short exact sequence

$$0 \longrightarrow Y^\bullet \xrightarrow{\nu_f} C(f)^\bullet \xrightarrow{\pi_f} X[1]^\bullet \longrightarrow 0$$

in $\mathsf{Com}^*(\mathcal{A})$. Since $\mathsf{Com}^*(\mathcal{A})$ is abelian, this induces a long exact sequence of cohomology objects

$$\cdots \longrightarrow H^i(Y^\bullet) \xrightarrow{\nu_f^i} H^i(C(f)^\bullet) \xrightarrow{\pi_f^i} H^i(X[1]^\bullet) \xrightarrow{\delta^i} H^{i+1}(Y^\bullet) \xrightarrow{\nu_f^{i+1}} H^{i+1}(C(f)^\bullet) \xrightarrow{\pi_f^{i+1}} H^{i+1}(X[1]^\bullet) \longrightarrow \cdots$$

where δ is the connecting morphism obtained from the Snake Lemma. Now fix a cohomology class $[c] \in H^i(X[1]^\bullet) = H^{i+1}(X^\bullet)$ for some cocycle $c \in X^{i+1}$. Then $\pi_f(0, c) = c$. On the other hand,

$$d_f(c,0) = \begin{pmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{pmatrix} \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} -d_X^{i+1}(c) \\ f^{i+1}(c) \end{pmatrix} = \begin{pmatrix} 0 \\ f^{i+1}(c) \end{pmatrix} = \nu_f^{i+1}(f^{i+1}(x))$$

Hence by the definition of the connecting morhism δ, we have that $\delta(\xi) = [f(x)]$ whence $\delta^i = f^i$. The above long exact sequence thus becomes

$$\cdots \longrightarrow H^i(Y^\bullet) \xrightarrow{\nu_f^i} H^i(C(f)^\bullet) \xrightarrow{\pi_f^i} H^i(X[1]^\bullet) \xrightarrow{f^{i+1}} H^{i+1}(Y^\bullet) \xrightarrow{\nu_f^{i+1}} H^{i+1}(C(f)^\bullet) \xrightarrow{\pi_f^{i+1}} H^{i+1}(X[1]^\bullet) \longrightarrow \cdots$$

But this is exactly the sequence obtained by applying the functor H^0 to the standard triangle

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{\nu_f} C(f)^\bullet \xrightarrow{\pi_f} X[1]^\bullet$$

and then extending in either direction using the axiom TR2. Hence $H^0 : \mathsf{K}^*(\mathcal{A}) \to \mathcal{A}$ is cohomological. $\square$

4.4 The Derived Category

Throughout this section, assume that $\mathcal{A}$ is an abelian category.

Definition 4.4.1. Let $f \in \mathsf{K}^*(\mathcal{A})(X^\bullet, Y^\bullet)$ be a morphism. We say that f is a **quasi-isomorphism** if it induces an isomorphism

$$H^q(f) : H^q(X^\bullet) \xrightarrow{\sim} H^q(Y^\bullet)$$

on cohomology objects. We denote by $\mathsf{Qis}^* = \mathsf{Qis}^*(\mathcal{A})$ the collection of all quasi-isomorphisms in $\mathsf{K}^*(\mathcal{A})$ for $* \in \{ \varnothing, +, -, b \}$.

Proposition 4.4.2. *Let $\mathcal{A}$ be an abelian category. Then Qis^* is a multiplicative system in $\mathsf{K}^*(\mathcal{A})$ and is compatible with the triangulation.*

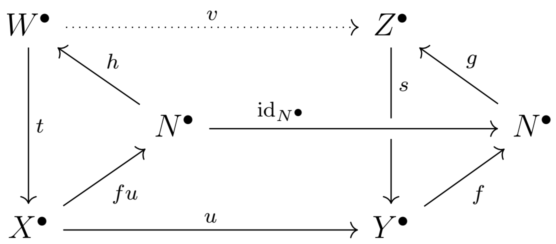
Proof.

<u>MS1:</u> It is immediate that the composition of two quasi-isomorphisms is again a quasi-isomorphism.

<u>MS2:</u> Fix a quasi-isomorphism $s : Z^\bullet \to Y^\bullet$ and suppose we are given a diagram

$$\begin{array}{ccc} & & Z^\bullet \\ & & \downarrow{\scriptstyle s} \\ X^\bullet & \xrightarrow{u} & Y^\bullet \end{array}$$

By TR1, we can embed s into a triangle $\Delta = (Z^\bullet, Y^\bullet, N^\bullet, s, f, g)$ and fu into a triangle $\Delta' = (W^\bullet, X^\bullet, N^\bullet, t, fu, h)$ so that we have a diagram



Then the tuple (u, id_N) is a map of two vertices of Δ' to Δ so that there exists a morphism $v : W^\bullet \to Z^\bullet$ such that $(v, u, \mathrm{id}_{N^\bullet})$ is a morphism of triangles. By construction we have that $sv = ut$ so it remains to show that t is a quasi-isomorphism.

By the long exact sequence of cohomology for Δ we have that $H^i(N^\bullet) = 0$ for all i since s induces an isomorphism $H^i(Z^\bullet) \to H^i(Y^\bullet)$ for all i. Inserting this into the long exact sequence of cohomology for Δ' then implies that t induces an isomorphism for all i so that t is a quasi-isomorphism.

The dual axiom to MS2 hold by dualising the above proof.

<u>MS3:</u> By additivity it suffices to show that, given $w \in \mathsf{K}^*(\mathcal{A})(X_1^\bullet, Y_1^\bullet)$, the following two statements are equivalent

- There exists $s \in \mathsf{K}^*(\mathcal{A})(Y_1^\bullet, Y_2^\bullet) \cap \mathsf{Qis}^*$ such that $sw = 0$.
- There exists $t \in \mathsf{K}^*(\mathcal{A})(X_2^\bullet, X_1^\bullet) \cap \mathsf{Qis}^*$ such that $wt = 0$.

We show that the first statement implies the second one. The reverse implication follows dually. By TR1, we can construct a triangle $\Delta = (Z^\bullet, Y_1^\bullet, Y_2^\bullet, x, s, y)$. By Proposition 2.2.1, we have a portion of a long exact sequence

$$\mathsf{K}^*(\mathcal{A})(X_1^\bullet, Z^\bullet) \xrightarrow{x^*} \mathsf{K}^*(\mathcal{A})(X_1^\bullet, Y_1^\bullet) \xrightarrow{s^*} \mathsf{K}^*(\mathcal{A})(X_1^\bullet, Y_2^\bullet)$$

Since $sw = 0$ it follows that $w \in \ker s^*$ and so there exists some $z \in \mathsf{K}^*(\mathcal{A})(X^\bullet, Z^\bullet)$ such that $x^*(z) = w$.

Appealing once more to TR1, we can construct a triangle $\Delta' = (X_2^\bullet, X_1^\bullet, Z^\bullet, t, z, m)$ leading to an exact sequence

$$\mathsf{K}^*(\mathcal{A})(Z^\bullet, Y_1^\bullet) \xrightarrow{z^*} \mathsf{K}^*(\mathcal{A})(X^\bullet, Y_1^\bullet) \xrightarrow{t^*} \mathsf{K}^*(\mathcal{A})(X_2^\bullet, Y_1^\bullet)$$

Since $w \in \operatorname{im} z^*$ we have that $t^*(w) = 0$ and so $wt = 0$. It remains to show that t is a quasi-isomorphism. By the long exact sequence of cohomology for the triangle Δ we know that $H^i(Z^\bullet) = 0$ for all i since s is a quasi-isomorphism. Inserting this into the long exact sequence of cohomology for the triangle Δ' then implies that t is a quasi-isomorphism.

MS4: It is immediate that $s \in \mathsf{Qis}^*$ if and only if $s[1] \in \mathsf{Qis}^*$ since shifting to the left by one position does not affect the fact that s induces an isomorphism on cohomology.

MS5: Let $(X_1^\bullet, Y_1^\bullet, Z_1^\bullet, \phi_1, \psi_1, \rho_1)$ and $(X_2^\bullet, Y_2^\bullet, Z_2^\bullet, \phi_2, \psi_2, \rho_2)$ be triangles in $\mathsf{K}^*(\mathcal{A})$. Given quasi isomorphisms $f : X_1^\bullet \to X_2^\bullet$ and $g : Y_1^\bullet \to Y_2^\bullet$, let $h : Z_1^\bullet \to Z_2^\bullet$ be the morphism completing the triple (f, g, h) to a morphism of triangles. We need to show that h is also a quasi isomorphism.

Since $H^0 : \mathsf{K}^*(\mathcal{A}) \to \mathcal{A}$ is a cohomological functor we have, for each $i \in \mathbb{Z}$, a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
H^i(X_1^\bullet) & \xrightarrow{\phi_1^i} & H^i(Y_1^\bullet) & \xrightarrow{\psi_1^i} & H^i(Z_1^\bullet) & \xrightarrow{\rho_1^i} & H^{i+1}(X_1^\bullet) & \xrightarrow{\phi_1^{i+1}} & H^{i+1}(Y_1^\bullet) \\
\wr\downarrow f^i & & \wr\downarrow g^i & & \downarrow h^i & & \wr\downarrow f^{i+1} & & \wr\downarrow g^{i+1} \\
H^i(X_2^\bullet) & \xrightarrow{\phi_2^i} & H^i(Y_2^\bullet) & \xrightarrow{\psi_2^i} & H^i(Z_2^\bullet) & \xrightarrow{\rho_2^i} & H^{i+1}(X_2^\bullet) & \xrightarrow{\phi_2^{i+1}} & H^i(Y_2'^\bullet)
\end{array}$$

By the 5-Lemma, we then have that h^i is an isomorphism for all i. Hence h is a quasi-isomorphism. □

Definition 4.4.3. We define the **derived category** $\mathsf{D}^*(\mathcal{A})$ of $\mathcal{A}$ to be the localisation

$$\mathsf{D}^*(\mathcal{A}) = (\mathsf{Qis}^*)^{-1}\mathsf{K}^*(\mathcal{A})$$

Theorem 4.4.4. $\mathsf{D}^*(\mathcal{A})$ *is a triangulated category. Moreover, the localisation functor* $Q : \mathsf{K}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{A})$ *is universal amongst additive* δ*-functors that map quasi-isomorphisms to isomorphisms.*

Proof. This follows from Theorem 4.3.8 and 3.4.2. □

4.5 Properties and Subcategories

Throughout this section, let $\mathcal{A}$ be an abelian category.

Proposition 4.5.1. $H^q : \mathsf{K}^*(\mathcal{A}) \to \mathcal{A}$ *extends uniquely to a functor* $H^q : \mathsf{D}^*(\mathcal{A}) \to \mathcal{A}$.

Proof. Let $s \in \mathsf{Qis}^*$ be a quasi-isomorphism. By definition, $H^q(s)$ is an isomorphism in $\mathcal{A}$. Hence H^q necessarily factors uniquely through the localisation functor $Q : \mathsf{K}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{A})$. By abuse of notation we denote this factorisation $H^q : \mathsf{D}^*(\mathcal{A}) \to \mathcal{A}$. □

Proposition 4.5.2. *Let* $\alpha \in \mathsf{D}^*(\mathcal{A})(X^\bullet, Z^\bullet)$ *be a morphism. Then* α *is an isomorphism if and only if it admits a representative roof* $s^{-1}f$ *such that* α *is a quasi-isomorphism.*

Proof. The backwards direction is clear by the definition of the localisation. Conversely, suppose that α is an isomorphism and let $s^{-1}f$ be a representative roof. Then $\alpha = Q(f) \circ Q(s)^{-1}$. $H^*(\alpha) = H^*(f) \circ H^*(s)^{-1}$. Since both $H^*(\alpha)$ and $H^*(s)^{-1}$ are isomorphisms in $\mathcal{A}$ it follows that so is $H^*(f)$. Then f is a quasi-isomorphism as claimed. □

Proposition 4.5.3. *Let* $f \in \mathsf{K}^*(\mathcal{A})(X^\bullet, Y^\bullet)$ *be a morphism. If* $Q(f) = 0$ *then the induced map on cohomology* f^* *is also* 0.

Proof. By Proposition 3.3.4, there exists a quasi-isomorphism $s : Z^\bullet \to X^\bullet$ such that $fs = 0$. Then $H^*(f) \circ H^*(s) = H^*(fs) = H^*(0) = 0$. But $H^*(s)$ is an isomorphism and so $H^*(f) = 0$. □

Proposition 4.5.4. *Denote by $\mathcal{B}$ the full subcategory of $\mathsf{D}(\mathcal{A})$ consisting of the complexes which are acyclic in all non-zero degrees. Then the canonical functor $F : \mathcal{A} \to \mathcal{B}$ induces an equivalence of categories.*

Proof. It suffices to show that F is fully faithful and essentially surjective. To this end, fix $X, Y \in \mathcal{A}$. By abuse of notation, identify X and Y with their images inside $\mathsf{D}(\mathcal{A})$. We need to show that the map

$$F : \mathcal{A}(X, Y) \to \mathsf{D}(\mathcal{A})(FX, FY)$$

of abelian groups is bijective. Suppose $u \in \mathcal{A}(X, Y)$ is a morphism such that $F(u) = 0$. Then $Q(u) = 0$. Proposition 4.5.3 then implies that the induced map on cohomology u^* is also zero. But $u^* = u$ since u is a morphism of complexes concentrated at 0. Hence the map is injective.

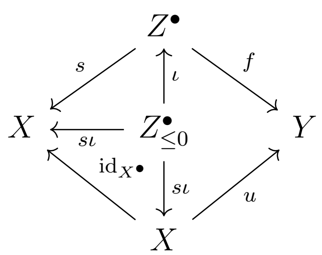
Now fix a morphism of complexes $v \in \mathsf{D}(\mathcal{A})(FX, FY)$ represented by a roof $s^{-1}f : FX \to Z^\bullet \to FY$. By hypothesis, we have an isomorphism $s^* : H^0(Z^\bullet) \to X$ and a morphism $f^* : H^0(Z^\bullet) \to Y$. Denote $u = f^*(s^*)^{-1}$. Denote by $Z^\bullet_{\leq 0}$ the complex obtained by truncated $Z^\bullet$ on positive degrees:

$$Z^i_{\leq 0} = \begin{cases} 0 & \text{if } i > 0 \\ \ker d^0 & \text{if } i = 0 \\ Z^i & \text{if } i < 0 \end{cases}$$

Then the canonical map $\iota : Z^\bullet_{\leq 0} \to Z^\bullet$ is a quasi-isomorphism and we have a commutative diagram

$$\begin{CD} Z^0_{\leq 0} @>{\iota}>> Z^0 \\ @VVV @VV{s}V \\ H^0(Z^\bullet) @>{s^*}>> X \end{CD}$$

And so we have a commutative diagram



which implies that $Q(u) = v$ and so F is fully faithful.

It remains to show that F is essentially surjective. Fix a complex $B^\bullet \in \operatorname{ob} \mathcal{B}$ that is acyclic in all non-zero degrees. We need to exhibit $A \in \operatorname{ob} \mathcal{A}$ such that $FA \cong B^\bullet$. Denote $A = H^0(B^\bullet)$ and $Z^\bullet = B^\bullet_{\leq 0}$. Then the canonical maps $\iota : Z^\bullet \to B^\bullet$ and $\rho : Z^\bullet \to FX$ are quasi-isomorphisms since $H^i(B^\bullet) = 0$ for all $i \neq 0$. Then the morphism $\alpha : FX \to B^\bullet$ represented by the roof $\rho^{-1}\iota$ is an isomorphism by Proposition 4.5.2. □

Proposition 4.5.5. *For all $* \in \{+, -, b\}$, the canonical functors $F^* : \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ are fully faithful.*

Proof. Suppose that $* = +$. The case of $* = -$ follows dually and for $* = b$ is a corollary of these two cases. By Proposition 3.3.2 we need to show that if $s \in \mathsf{K}(X^\bullet, Y^\bullet) \cap \mathsf{Qis}$, where $X^\bullet \in \operatorname{ob} \mathsf{K}^+(\mathcal{A})$, then there exists $Z^\bullet \in \operatorname{ob} \mathsf{K}^+(\mathcal{A})$ and a morphism $f : Y^\bullet \to Z^\bullet$ such that $fs \in \mathsf{Qis}$.

Clearly, we can assume that $X^i = 0$ for all $i < 0$. Since s induces an isomorphism on cohomology objects, it follows that $H^i(Y^\bullet) = 0$ for all $i < 0$. Now denote

$$Z^i = \begin{cases} 0 & \text{if } i < 0 \\ \operatorname{coker}(d : Y^{-1} \to Y^0) & \text{if } i = 0 \\ Y^i & \text{if } i > 0 \end{cases}$$

Then $Z^\bullet \in \operatorname{ob} \mathsf{K}^+(\mathcal{A})$ and the obvious morphism $Y^\bullet \to Z^\bullet$ is clearly a quasi-isomorphism. $\square$

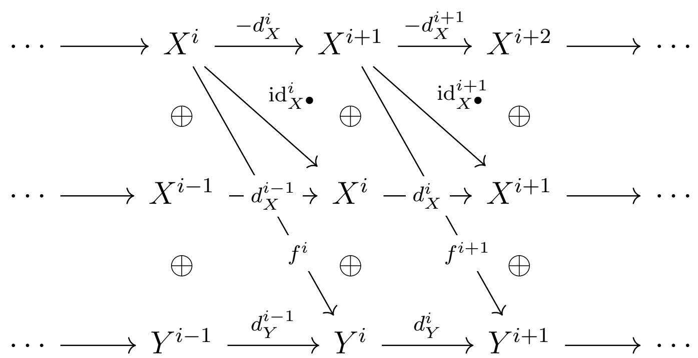
4.6 Short Exact Sequences and the Mapping Cylinder

Throughout this section we shall assume that $\mathcal{A}$ is an abelian category.

Definition 4.6.1. Let $f \in \mathsf{Com}^*(\mathcal{A})(X^\bullet, Y^\bullet)$ be a morphism. We define the **mapping cylinder** of f to be the complex given by the data

$$\operatorname{Cyl}(f)^\bullet = X[1]^\bullet \oplus X^\bullet \oplus Y^\bullet$$
$$d^i_{\operatorname{Cyl}(f)} = \begin{pmatrix} -d_X^{i+1} & 0 & 0 \\ -\operatorname{id}^{i+1}_{X^\bullet} & d_X^i & 0 \\ f^{i+1} & 0 & d_Y^i \end{pmatrix}$$

The mapping cylinder can be visualised as a diagram



The canonical projection $\kappa_f : \operatorname{Cyl}(f)^\bullet \to C(f)^\bullet$ and inclusion $\iota_f : X^\bullet \to \operatorname{Cyl}(f)^\bullet$ induce a canonical exact sequence

$$0 \longrightarrow X^\bullet \xrightarrow{\iota_f} \operatorname{Cyl}(f)^\bullet \xrightarrow{\kappa_f} C(f)^\bullet \longrightarrow 0$$

of complexes in $\mathsf{Com}(\mathcal{A})$.

Lemma 4.6.2. *Let $f \in \mathsf{Com}^*(\mathcal{A})(X^\bullet, Y^\bullet)$ be a morphism. Then there exist morphisms $\sigma : Y^\bullet \to \operatorname{Cyl}(f)^\bullet$ given by $\sigma^i = (0, 0, \operatorname{id}^i_{Y^\bullet})$ and $\tau : \operatorname{Cyl}(f)^\bullet \to Y^\bullet$ given by $\tau^i = (0, f^i, \operatorname{id}^i_{Y^\bullet})$ such that*

1. *The diagram with exact rows*

$$
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & Y^\bullet & \xrightarrow{\nu_f} & C(f)^\bullet & \xrightarrow{\pi_f} & X[1]^\bullet \longrightarrow 0 \\
 & & & & \big\downarrow \sigma & & \big\downarrow \mathrm{id}_{X^\bullet} & & \\
0 & \longrightarrow & X^\bullet & \xrightarrow{\iota_f} & \mathrm{Cyl}(f)^\bullet & \xrightarrow{\kappa_f} & C(f)^\bullet & \longrightarrow & 0 \\
 & & \big\downarrow \mathrm{id}_{X^\bullet} & & \big\downarrow \tau & & & & \\
 & & X^\bullet & \xrightarrow{f} & Y^\bullet & & & &
\end{array}
$$

is commutative.

2. $\tau \circ \sigma = \mathrm{id}_{Y^\bullet}$ *and* $\sigma \circ \tau$ *is homotopic to* $\mathrm{id}_{\mathrm{Cyl}(f)^\bullet}$*. In particular,* $Y^\bullet$ *and* $\mathrm{Cyl}(f)^\bullet$ *are isomorphic in* $\mathsf{K}^*(\mathcal{A})$ *and thus in* $\mathsf{D}^*(\mathcal{A})$*.*

3. σ *and* τ *are quasi-isomorphisms.*

Proof.

<u>Part 1:</u> It is evident from the definitions that the two squares in the above diagram commute

<u>Part 2:</u> It is clear that $\tau \circ \sigma = \mathrm{id}_{Y^\bullet}$. We claim that the map given by the matrix

$$k^i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{id}^i_{X^\bullet} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a homotopy from $\sigma \circ \tau$ to $\mathrm{id}_{\mathrm{Cyl}(f)^\bullet}$. Indeed, $d^{i-1}_{\mathrm{Cyl}(f)} k^i + k^{i+1} d^i_{\mathrm{Cyl}(f)}$ is given by

$$
\begin{aligned}
&\begin{pmatrix} -d^i_X & 0 & 0 \\ -\mathrm{id}^i_{X^\bullet} & d^{i-1}_X & 0 \\ f^i & 0 & d^{i-1}_Y \end{pmatrix}
\begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{id}^i_{X^\bullet} & 0 \\ 0 & 0 & 0 \end{pmatrix}
+
\begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{id}^{i+1}_{X^\bullet} & 0 \\ 0 & 0 & 0 \end{pmatrix}
\begin{pmatrix} -d^{i+1}_X & 0 & 0 \\ -\mathrm{id}^{i+1}_{X^\bullet} & d^i_X & 0 \\ f^{i+1} & 0 & d^i_Y \end{pmatrix} \\
&= \begin{pmatrix} 0 & -d^i_X \mathrm{id}^i_{X^\bullet} & 0 \\ 0 & -\mathrm{id}^i_{X^\bullet} & 0 \\ 0 & f^i & 0 \end{pmatrix}
+ \begin{pmatrix} -\mathrm{id}^{i+1}_{X^\bullet} & \mathrm{id}^{i+1}_{X^\bullet} d^i_X & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\mathrm{id}^{i+1}_{X^\bullet} & 0 & 0 \\ 0 & -\mathrm{id}^i_{X^\bullet} & 0 \\ 0 & f^i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & f^i & \mathrm{id}^i_{Y^\bullet} \end{pmatrix}
- \begin{pmatrix} \mathrm{id}^{i+1}_{X^\bullet} & 0 & 0 \\ 0 & \mathrm{id}^i_{X^\bullet} & 0 \\ 0 & 0 & \mathrm{id}^i_{Y^\bullet} \end{pmatrix} \\
&= \sigma^i \circ \tau^i - \mathrm{id}^i_{\mathrm{Cyl}(f)^\bullet}
\end{aligned}
$$

<u>Part 3:</u> By Part 2, $\sigma \circ \tau$ and $\tau \circ \sigma$ are both homotopic to the identity maps and, so, induce the identity map on cohomology. Hence

$$H^i(\tau) \circ H^i(\sigma) = H^i(\tau \circ \sigma) = H^i(\mathrm{id}_{\mathrm{Cyl}(f)^\bullet}) = \mathrm{id}_{H^i(\mathrm{Cyl}(f)^\bullet)}$$

Similarly, $H^i(\sigma) \circ H^i(\tau) = \mathrm{id}_{H^i(Y^\bullet)}$ and so $H^i(\sigma)$ and $H^i(\tau)$ are isomorphisms whence σ and τ are quasi-isomorphisms.

□

Proposition 4.6.3. *Let*

$$0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0$$

be a short exact sequence in $\mathsf{Com}^*(\mathcal{A})$*. Then there exists a triangle*

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow X[1]^\bullet$$

in $\mathsf{D}^*(\mathcal{A})$.

Proof. Consider the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X^\bullet & \xrightarrow{\iota} & \mathrm{Cyl}(f)^\bullet & \xrightarrow{\kappa_f} & C(f)^\bullet & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}_{X^\bullet}} & & \downarrow{\scriptstyle \tau} & & \vdots{\scriptstyle \gamma} & & \\
0 & \longrightarrow & X^\bullet & \xrightarrow{f} & Y^\bullet & \xrightarrow{g} & Z^\bullet & \longrightarrow & 0
\end{array}$$

Define a morphism of complexes $\gamma : C(f)^\bullet \to Z^\bullet$ by $\gamma^i(x,y) = g^i(y)$. It is then clear that the right hand square in the above diagram commutes. Now, $\mathrm{id}_{X^\bullet}$ and τ are both quasi-isomorphisms by the previous Lemma. The long exact sequence of cohomology together with the 5-Lemma then imply that γ is itself a quasi-isomorphism. In particular, $\mathrm{id}_{X^\bullet}, \tau$ and γ are all isomorphisms in $\mathsf{D}^*(\mathcal{A})$. Now consider the diagram

$$\begin{array}{ccccccc}
X^\bullet & \xrightarrow{f} & Y^\bullet & \xrightarrow{\nu_f} & C(f)^\bullet & \xrightarrow{\pi_f} & X[1]^\bullet \\
\downarrow{\scriptstyle \mathrm{id}_{X^\bullet}} & & \downarrow{\scriptstyle \mathrm{id}_{Y^\bullet}} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{id}_{X[1]^\bullet}} \\
X^\bullet & \xrightarrow{f} & Y^\bullet & \xrightarrow{g} & Z^\bullet & \xrightarrow{\nu_f \circ \gamma^{-1}} & X[1]^\bullet
\end{array}$$

in $\mathsf{D}^*(\mathcal{A})$. For the commutativity of the second square, note that by the Lemma and the definition of γ we have

$$\gamma \circ \nu_f = \gamma \circ \kappa_f \circ \sigma = g \circ \tau \circ \sigma = g \circ \mathrm{id}_{Y^\bullet} = g$$

Hence this diagram is an isomorphism of triangles in $\mathsf{D}^*(\mathcal{A})$ and so

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \xrightarrow{\nu_f \circ \gamma^{-1}} X[1]^\bullet$$

is the desired triangle of the Proposition. □

5 Derived Functors

Throughout this section we assume that $\mathcal{A}$ and $\mathcal{B}$ are abelian categories. We denote by $\mathsf{Qis}_\mathcal{A}$ and $\mathsf{Qis}_\mathcal{B}$ the collections of quasi-isomorphisms in $\mathsf{K}^*(\mathcal{A})$ and $\mathsf{K}^*(\mathcal{B})$ respectively. Moreover, let $Q_\mathcal{A} : \mathsf{K}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{A})$ and $Q_\mathcal{B} : \mathsf{K}^*(\mathcal{B}) \to \mathsf{D}^*(\mathcal{B})$ be the localisation functors.

5.1 Exact Functors

Proposition 5.1.1. *Let* $F : \mathcal{A} \to \mathcal{B}$ *be an exact functor. Then*

1. F *extends naturally to a functor* $F^* : \mathsf{Com}^*(\mathcal{A}) \to \mathsf{Com}^*(\mathcal{B})$ *which preserves homotopy equivalences.*

2. F^* *descends to a functor* $F^* : \mathsf{K}^*(\mathcal{A}) \to \mathsf{K}^*(\mathcal{B})$ *such that* $F^*(\mathsf{Qis}_\mathcal{A}) \subseteq \mathsf{Qis}_\mathcal{B}$.

3. F^* *extends to a* δ*-functor* $F^* : \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{B})$.

Proof.

Part 1: Let $F^* : \mathsf{Com}^*(\mathcal{A}) \to \mathsf{Com}^*(\mathcal{B})$ be given component-wise by F. That F^* enjoys the properties of a functor is immediate from the fact that F does. Now suppose that $f, g \in \mathsf{Com}^*(X^\bullet, Y^\bullet)$ are morphisms which are homotopic via $k \in \mathcal{A}^{-1}(X^\bullet, Y^\bullet)$. Then it is clear that $F(f)$ is homotopic to $F(g)$ via the homotopy $F(k)$.[2]

Part 2: Since $F^* : \mathsf{Com}^*(\mathcal{A}) \to \mathsf{Com}^*(\mathcal{B})$ is constant on homotopy classes of morphisms in $\mathsf{Com}^*(\mathcal{A})$, it descends to a functor $F^* : \mathsf{K}^*(\mathcal{A}) \to \mathsf{K}^*(\mathcal{B})$. To show that $F^*(\mathsf{Qis}_\mathcal{A}) \subseteq \mathsf{Qis}_\mathcal{B}$, we first claim that F^* preserves acyclic complexes.

Fix an acyclic complex $K^\bullet$ in $\mathsf{K}^*(\mathcal{A})$. Denote $B^i = \ker d_K^i = \operatorname{im} d_K^{i-1}$. Then F maps the short exact sequence

$$0 \longrightarrow B^i \xrightarrow{e^i} K^i \xrightarrow{p^i} B^{i+1} \longrightarrow 0$$

to a short exact sequence

$$0 \longrightarrow F(B^i) \xrightarrow{F(e^i)} F(K^i) \xrightarrow{F(p^i)} F(B^{i+1}) \longrightarrow 0$$

Since $d_K^i = e^{i+1} \circ p^i$, we have that $F(d_K^i) = F(e^{i+1}) \circ F(p^i)$. But $F(e^{i+1})$ is an monomorphism and $F(p^i)$ is an epimorphism. Hence $F(B^i)$ is isomorphic to the image of d_K^{i-1} and $F(B^{i+1})$ is isomorphic to the cokernel of $F(d_K^{i+1})$. This is precisely what it means for the complex $F^*(K^\bullet)$ to be acyclic.

Now observe that, given a morphism $f \in \mathsf{K}^*(X^\bullet, Y^\bullet)$, the long exact sequence of cohomology implies that f is a quasi-isomorphism if and only if the mapping cone $C(f)$ is acyclic. Then, by the above argumentation,

$$\begin{aligned} f \in \mathsf{Qis}_\mathcal{A} &\iff C(f) \text{ is acyclic} \\ &\implies F^*(C(f)) \text{ is acyclic} \\ &\iff C(F^*(f)) \text{ is acyclic} \qquad (F^*(C(f)) \cong C(F^*(f))) \\ &\iff F^*(f) \in \mathsf{Qis}_\mathcal{B} \end{aligned}$$

Part 3: Consider the composition $Q_\mathcal{B} \circ F^* : \mathsf{K}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{B})$. Then $Q_\mathcal{B} \circ F^*$ maps $\mathsf{Qis}_\mathcal{A}$ to isomorphisms in $\mathsf{D}^*(\mathcal{B})$. By the universal property of $Q_\mathcal{A}$, there exists a unique functor $F^* : \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{B})$ such that the diagram

$$\begin{array}{ccc} \mathsf{K}^*(\mathcal{A}) & \xrightarrow{F^*} & \mathsf{K}^*(\mathcal{B}) \\ \downarrow{\scriptstyle Q_\mathcal{A}} & & \downarrow{\scriptstyle Q_\mathcal{B}} \\ \mathsf{D}^*(\mathcal{A}) & \xrightarrow{F^*} & \mathsf{D}^*(\mathcal{B}) \end{array}$$

commutes. It remains to show that $F^* : \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{B})$ is a δ-functor. Since there is an isomorphism $F(C(f)) \cong C(F(f))$, it follows that the image of a standard triangle in $\mathsf{K}^*(\mathcal{A})$ under F^* is a standard triangle in $\mathsf{K}^*(\mathcal{B})$. By Part 2, $F^*(\mathsf{Qis}_\mathcal{A}) \subseteq \mathsf{Qis}_\mathcal{B}$ so that any diagram in $\mathsf{K}^*(\mathcal{A})$ that is quasi-isomorphic to a standard triangle has its images under F^* also quasi-isomorphic to a standard triangle. Passing to the derived categories, we see that any diagram in $\mathsf{D}^*(\mathcal{A})$ that is isomorphic to a standard triangle has its image under F^* also isomorphic to a standard triangle. Hence F^* maps triangles in $\mathsf{D}^*(\mathcal{A})$ to triangles in $\mathsf{D}^*(\mathcal{B})$ whence F^* is a δ-functor. □

[2]Note that here we did not use the full fact that F is exact - merely that it is additive.

5.2 Resolutions

Definition 5.2.1. We say that a collection $\mathfrak{R} = \mathfrak{R}(\mathcal{A}) \subseteq \operatorname{ob}\mathcal{A}$ of objects is **admissible** if it contains the zero object and is closed under direct sums and isomorphisms. We denote by $\mathsf{Com}^*(\mathfrak{R})$ (resp. $\mathsf{K}^*(\mathfrak{R})$) the full subcategory of $\mathsf{Com}^*(\mathcal{A})$ (resp. $\mathsf{K}^*(\mathfrak{R})$) consisting of complexes whose every component is an object of $\mathfrak{R}$.

Definition 5.2.2. Let $\mathfrak{R}$ be an admissible collection of objects of $\mathcal{A}$. We say that $\mathcal{A}$ has **enough right $\mathfrak{R}$-objects** (resp. **left $\mathfrak{R}$-objects**) if for every $A \in \operatorname{ob}\mathcal{A}$ there exists an $R \in \mathfrak{R}$ and a monomorphism $A \to R$ (resp. an epimorphism $R \to A$). Moreover, if $\mathcal{A}$ has enough left and right $\mathfrak{R}$-objects we say that $\mathcal{A}$ has **enough two-sided $\mathfrak{R}$-objects**.

Definition 5.2.3. Let $A^\bullet \in \operatorname{ob}\mathsf{Com}^*(\mathcal{A})$ be a complex. We define a **right $\mathfrak{R}$-resolution** (resp. **left $\mathfrak{R}$-resolution**) of $A^\bullet$ to be a complex $R^\bullet \in \mathsf{Com}^*(\mathfrak{R})$ together with a quasi-isomorphism $A^\bullet \to R^\bullet$ (resp. $R^\bullet \to A^\bullet$).

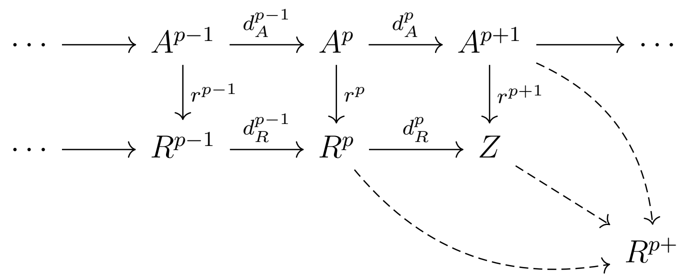
Remark. Let $\mathcal{A}$ be an abelian category with enough injectives and let $\mathfrak{R}$ be the collection of injective objects of $\mathcal{A}$. Then $\mathfrak{R}$ is admissible. Consider the complex concentrated at degree 0 $A[0]$. Then a right $\mathfrak{R}$-resolution of $A[0]$ is a complex $X^\bullet$ whose every component is an injective object of $\mathcal{A}$, $X^i = 0$ for $i < 0$ together with a quasi-isomorphism $A[0]^\bullet \to X^\bullet$. This is just the usual definition of an injective resolution of $\mathcal{A}$.

Proposition 5.2.4. *Let $\mathfrak{R}$ be an admissible collection of objects of $\mathcal{A}$ and let $X^\bullet \in \mathsf{Com}^+(\mathcal{A})$ (resp. $* = -$) be a complex. If $R^\bullet$ and $R'^\bullet$ are two right (resp. left) $\mathfrak{R}$-resolutions of $X^\bullet$ then $R^\bullet$ and $R'^\bullet$ are isomorphic in $\mathsf{D}^+(\mathcal{A})$ (resp. $* = -$).*

Proof. This is immediate from the fact that resolutions are quasi-isomorphisms which become isomorphisms in $\mathsf{D}^+(\mathcal{A})$. □

Proposition 5.2.5. *Let $\mathfrak{R}$ be an admissible collection of objects of $\mathcal{A}$ and suppose that $\mathcal{A}$ has sufficiently many right (resp. left) $\mathfrak{R}$-objects. Then every object in $\mathsf{Com}^+(\mathcal{A})$ (resp. $\mathsf{Com}^-(\mathcal{A})$ admits a monic right (resp. epic left) $\mathfrak{R}$-resolution.*

Proof. Fix a complex $A^\bullet \in \operatorname{ob}\mathsf{Com}^+(\mathcal{A})$. We will exhibit a complex $R^\bullet \in \operatorname{ob}\mathsf{Com}^+(\mathfrak{R})$ together with a monomorphism that is a quasi-isomorphism $r : A^\bullet \to R^\bullet$. Without loss of generality, we may assume that $A^i = 0$ for all $i < 0$. We construct $R^\bullet$ by induction on the degree of the complex. Suppose that we have constructed R^i up to degree $p \in \mathbb{Z}^+$ and that we have monomorphisms r^q for $q \leq p$ which are quasi-isomorphisms for $q \leq p-1$. Consider the diagram



where Z is the apex of the pushforward of the morphisms d_A^p and $\operatorname{coker}(d_R^{p-1}) \circ r^p$.[3] It follows immediately from the definition of the pushforward that $d_R^p \circ r^p = r^{p+1} \circ d_A^p$. Moreover, the fact that d_R^p factors through the cokernel of d_R^{p-1} implies that $d_R^p \circ d_R^{p-1} = 0$.

[3]Note that, using Mitchell's Embedding Theorem, we can interpret this cokernel as the projection map $\pi_p : R^p \to R^p / \operatorname{im} d_R^{p-1}$.

Recall that, explicitly, the pushout Z is given by $Z = (R^p / \operatorname{im} d_R^{p-1} \oplus A^{p+1})/M$ where M is the subobject generated by the relations

$$((\pi_p \circ r^p)(a), 0) \sim (0, -d_A^p(a))$$

for all $a \in A^p$. Using this description, we show that r^{p+1} is monic. Fix $x \in A^{p+1}$ such that $r^{p+1}(x) = (0, x) = 0$. Then there exists $a \in A^p$ such that $(0, x) = ((\pi_p \circ r^p)(a), d_A^p(a))$. Hence $x = d_A^p(a)$ and there exists some $y \in R^{p-1}$ such that $r^p(a) = d_R^{p-1}(y)$.

Now, r^{p-1} is a quasi-isomorphism. Hence we can always find $u \in A^{p-1}$ such that we have the equality of cohomology classes $[y] = [r^{p-1}(u)]$. In particular, $y - r^{p-1}(u) \in \operatorname{im} d_R^{p-2}$ and so there exists some $z \in R^{p-2}$ such that $y - r^{p-1}(u) = d_R^{p-2}(z)$. Applying d_R^{p-1} to both sides of this equation, we thus see that

$$r^p(a) = d_R^{p-1}(y) = (d_R^{p-1} \circ r^{p-1})(u) = (r^p \circ d_A^{p-1})(u)$$

But r^p is monic and so $a = d_A^{p-1}(u)$. It then follows that $x = d_A^p(a) = (d_A^p \circ d_A^{p-1})(u) = 0$ and so r^{p+1} is monic.

We now show that $(r^p)^*$ is an isomorphism. To this end, fix $[c] \in H^p(A^\bullet)$ and suppose that $(r^p)^*([c]) = 0$. Then $[r^p(c)] = 0$ which is to say $r^p(c) \in \operatorname{im} d_R^{p-1}$. Hence there exists $y \in I^{p-1}$ such that $r^p(c) = d_R^{p-1}(y)$. As before, this implies that there exists $u \in A^{p-1}$ such that $[y] = [r^{p-1}(u)]$ and $d_A^{p-1}(u) = c$. Hence $[c] = 0$ and $(r^p)^*$ is injective. To see that $(r^p)^*$ is surjective, fix $[y] \in H^p(R^\bullet)$. Then $y \in \ker d_R^p \subseteq \operatorname{im} d_R^{p-1}$ so that there exists $z \in R^{p-1}$ such that $d_R^{p-1}(z) = y$. Since r^{p-1} is a quasi-isomorphism, there exists $u \in A^{p-1}$ such that $[z] = [r^{p-1}(u)]$. Then

$$y = d_R^{p-1}(z) = (d_R^{p-1} \circ r^{p-1})(u) = (r^p \circ d_R^{p-1})(u)$$

which implies that $[y] = [r^p(d_R^{p-1}(u))] = (r^p)^*([d_R^{p-1}(u)])$ so that $(r^p)^*$ is surjective.

Finally, we have, by hypothesis, an object $R^{p+1} \in \mathfrak{R}$ together with a monomomorphism $Z \to R^{p+1}$. By abuse of notation, redefine r^{p+1} and d_R^p to be the composition of r^{p+1} and d_R^p respectively with this monomorphism. This then gives us a monic right $\mathfrak{R}$-resolution $A^\bullet \to R^\bullet$. For the case of left $\mathfrak{R}$-objects, we can simply dualise this argument to produce an epic left $\mathfrak{R}$-resolution $R^\bullet \to A^\bullet$. □

Corollary 5.2.6. *Let $\mathfrak{R}$ be an admissible class and $\mathsf{Qis}^*_{\mathfrak{R}}$ the collection of quasi-isomorphisms in $\mathsf{K}^*(\mathfrak{R})$. Then*

1. *$\mathsf{Qis}^*_{\mathfrak{R}}$ is a multiplicative system.*
2. *If $\mathcal{A}$ has enough right (resp. left, two-sided) $\mathfrak{R}$-objects then the natural functor*

$$F : (\mathsf{Qis}^*_{\mathfrak{R}})^{-1}\mathsf{K}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{A})$$

induces an equivalence between $(\mathsf{Qis}^+_{\mathfrak{R}})^{-1}\mathsf{K}^+(\mathfrak{R})$ (resp. $* \in \{-, b\}$) and $\mathsf{D}^*(\mathcal{A})$.

Proof. That $\mathsf{Qis}^*_{\mathfrak{R}}$ is a multiplicative system in $\mathsf{K}^*(\mathfrak{R})$ follows the same proof as Proposition 4.4.2. It suffices to observe that this explicit construction involved passing to mapping cones of certain morphisms. Since $\mathfrak{R}$ is admissible, direct sums of $\mathfrak{R}$-objects are $\mathfrak{R}$-objects. In particular, the mapping cone of a morphism in $\mathsf{K}^*(\mathfrak{R})$ is again in $\mathsf{K}^*(\mathfrak{R})$. Hence the proof of Proposition 4.4.2 follows through in this case as well.

Now suppose that $\mathcal{A}$ has enough right $\mathfrak{R}$-objects. We first claim that F is fully faithful. Observe that, by Proposition 3.3.2, it suffices to show that if $s : R^\bullet \to X^\bullet$ is a quasi-isomorphism in Qis^+ with $R^\bullet \in \mathsf{K}^+(\mathfrak{R})$ then there exists a quasi-isomorphism $t : X^\bullet \to R'^\bullet$

with $R'^\bullet \in \mathsf{K}^+(\mathfrak{R})$ such that $t \circ s \in \mathsf{Qis}^+_{\mathfrak{R}}$. Appealing to the Proposition, we can construct a right $\mathfrak{R}$-resolution $t : X^\bullet \to R'^\bullet$ and it is clear that $t \circ s \in \mathsf{Qis}^+_{\mathfrak{R}}$ and so F induces an equivalence onto a full subcategory of $\mathsf{D}^+(\mathcal{A})$. Dualising this argument proves the case of $* = -$ and the case of $* = b$ follows by combining these two results.

To show that F is an equivalence onto $\mathsf{D}^+(\mathcal{A})$, it suffices to show that F is essentially surjective. To this end, fix a complex $X^\bullet \in \mathsf{D}^+(\mathcal{A})$. By the Proposition, we can construct a right $\mathfrak{R}$-resolution $X^\bullet \to R^\bullet$ for some $R^\bullet \in \mathsf{K}^+(\mathfrak{R})$. Then $F(R^\bullet) \cong X^\bullet$ in $\mathsf{D}^+(\mathcal{A})$ so that F is indeed essentially surjective. Dualising this argument proves the case of $* = -$. Now suppose that $* = b$. Fix a bounded complex $X^\bullet \in \mathsf{D}^b(\mathcal{A})$. Then we can construct a right $\mathfrak{R}$-resolution $s : X^\bullet \to R^\bullet$ and then a left $\mathfrak{R}$-resolution $t : R'^\bullet \to R^\bullet$. Clearly, $R'^\bullet$ is a bounded complex which is isomorphic to $X^\bullet$ in $\mathsf{D}^b(\mathcal{A})$ so that F is also essentially surjective in this case. □

5.3 Derived Functors

Throughout this section, let $\mathfrak{R}$ be an admissible class of objects of $\mathcal{A}$.

Definition 5.3.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a left (resp. right) exact functor. We say that $\mathfrak{R}$ is **adapted** to F if $F^+ : \mathsf{Com}^+(\mathfrak{R}) \to \mathsf{Com}^+(\mathfrak{R})$ (resp. $* = -$) maps acyclic complexes to acyclic complexes.

Definition 5.3.2. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact covariant functor. We define the **right derived functor** of F (should it exist) to be a pair $(\mathsf{R}F, \varepsilon_F)$ where $\mathsf{R}F : \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$ is a δ-functor and $\varepsilon_F : Q_{\mathcal{B}} \circ F^+ \to \mathsf{R}F \circ Q_{\mathcal{A}}$ is a natural transformation in the context of the following diagram

$$\begin{array}{ccc} \mathsf{K}^+(\mathcal{A}) & \xrightarrow{F^+} & \mathsf{K}^+(\mathcal{B}) \\ \downarrow{\scriptstyle Q_{\mathcal{A}}} & & \downarrow{\scriptstyle Q_{\mathcal{B}}} \\ \mathsf{D}^+(\mathcal{A}) & \xrightarrow{\mathsf{R}F} & \mathsf{D}^+(\mathcal{B}) \end{array}$$

which is universal amongst pairs (G, ε). To be precise, given any δ-functor $G : \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$ and a natural transformation $\varepsilon : Q_{\mathcal{B}} \circ F^+ \to G \circ Q_{\mathcal{A}}$, there exists a unique natural transformation $\eta : \mathsf{R}F \to G$ such that the diagram

$$\begin{array}{ccc} & Q_{\mathcal{B}} \circ F^+ & \\ {\scriptstyle \varepsilon_F} \swarrow & & \searrow {\scriptstyle \varepsilon} \\ \mathsf{R}F \circ Q_{\mathcal{A}} & \xrightarrow{\eta_{Q_{\mathcal{A}}}} & G \circ Q_{\mathcal{A}} \end{array}$$

commutes.

Remark.

1. Given a right exact covariant functor $F : \mathcal{A} \to \mathcal{B}$ we define dually the **left derived functor** $(\mathsf{L}F, \varepsilon_F)$ where $\mathsf{L}F : \mathsf{D}^-(\mathcal{A}) \to \mathsf{D}^-(\mathcal{B})$ is a δ-functor and $\varepsilon_F : \mathsf{L}F \circ Q_{\mathcal{A}} \to Q_{\mathcal{B}} \circ F^-$ is a natural transformation which is universal amongst pairs (G, ε) as above.

2. If F is a (right or left exact) contravariant functor then we swap the definitions of right and left derived functors around.

3. Suppose that we are given a δ-functor $F : \mathsf{K}^+(\mathcal{A}) \to \mathsf{K}^+(\mathcal{B})$. Then we can make the same definitions for the right and left derived functors of F.

From now on, we shall only deal with covariant left-exact functors $F : \mathcal{A} \to \mathcal{B}$. Everything follows through for the other cases in the Remark above by making the appropriate modifications.

Lemma 5.3.3. *Suppose that $\mathcal{A}$ has enough right $\mathfrak{R}$-objects. Let Δ be a diagram*

$$X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow Z^\bullet \longrightarrow X[1]^\bullet$$

in $(\mathsf{Qis}^+_{\mathfrak{R}})^{-1}\mathsf{K}(\mathfrak{R})$ that becomes a triangle in $\mathsf{D}^+(\mathcal{A})$. Then Δ is isomorphic in $\mathsf{D}^+(\mathcal{A})$ to a triangle in $\mathsf{K}^+(\mathfrak{R})$.

Proof. By Corollary 5.2.6, we have an equivalence of categories

$$F : (\mathsf{Qis}^+_{\mathfrak{R}})^{-1}\mathsf{K}(\mathfrak{R}) \to \mathsf{D}^+(\mathcal{A})$$

Hence $f \in \mathsf{D}^+(\mathcal{A})(X^\bullet, Y^\bullet)$ admits a representation as a roof $s^{-1}r : X^\bullet \to T^\bullet \to Y^\bullet$ where $T^\bullet \in \mathsf{K}^+(\mathfrak{R})$ and $q \in \mathsf{Qis}^+_{\mathfrak{R}}$. We claim that Δ is isomorphic in $\mathsf{D}^+(\mathcal{A})$ to the standard triangle

$$T^\bullet \xrightarrow{r} Y^\bullet \longrightarrow C(r)^\bullet \longrightarrow T[1]^\bullet$$

in $\mathsf{K}^+(\mathfrak{R})$. Indeed, consider the diagram

$$\begin{array}{ccccccc}
T^\bullet & \xrightarrow{r} & Y^\bullet & \longrightarrow & C(r)^\bullet & \longrightarrow & T[1]^\bullet \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \mathrm{id}_{Y^\bullet}} & & \vdots{\scriptstyle v} & & \downarrow{\scriptstyle q[1]} \\
X^\bullet & \xrightarrow{f} & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & X[1]^\bullet
\end{array}$$

The two rows are triangles in $\mathsf{D}^+(\mathcal{A})$ and the left-hand square is commutative. Hence there exists a morphism $v : C(r)^\bullet \to Z^\bullet$ completing the diagram to a morphism of triangles. But q and $\mathrm{id}_{Y^\bullet}$ are isomorphisms whence v is also an isomorphism. Hence the above diagram is an isomorphism of triangles in $\mathsf{D}^+(\mathcal{A})$. □

Theorem 5.3.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor. Suppose that $\mathcal{A}$ has enough right $\mathfrak{R}$-objects and that $\mathfrak{R}$ is adapted for F. Then $\mathsf{R}F$ exists and is unique up to unique natural isomorphism.*

Proof.

Uniqueness: Suppose that $(\mathsf{R}F, \varepsilon_F)$ and $(\mathsf{R}'F, \varepsilon'_F)$ are two right derived functors of F. Then there exist unique natural transformations $\eta' : \mathsf{R}'F \to \mathsf{R}F$ and $\eta : \mathsf{R}F \to \mathsf{R}'F$ such that the diagrams

$$\begin{array}{ccc}
 & Q_{\mathcal{B}} \circ F^+ & \\
\swarrow{\scriptstyle \varepsilon'_F} & & \searrow{\scriptstyle \varepsilon_F} \\
\mathsf{R}'F \circ Q_{\mathcal{A}} & \xrightarrow{\eta'_{Q_{\mathcal{A}}}} & \mathsf{R}F \circ Q_{\mathcal{A}}
\end{array}
\qquad
\begin{array}{ccc}
 & Q_{\mathcal{B}} \circ F^+ & \\
\swarrow{\scriptstyle \varepsilon_F} & & \searrow{\scriptstyle \varepsilon'_F} \\
\mathsf{R}F \circ Q_{\mathcal{A}} & \xrightarrow{\eta_{Q_{\mathcal{A}}}} & \mathsf{R}'F \circ Q_{\mathcal{A}}
\end{array}$$

commute. This implies that the natural transformation $\eta \circ \eta'$ makes the diagram

$$\begin{array}{ccc}
 & Q_{\mathcal{B}} \circ F^+ & \\
\swarrow{\scriptstyle \varepsilon'_F} & & \searrow{\scriptstyle \varepsilon'_F} \\
\mathsf{R}'F \circ Q_{\mathcal{A}} & \xrightarrow{(\eta \circ \eta')_{Q_{\mathcal{A}}}} & \mathsf{R}'F \circ Q_{\mathcal{A}}
\end{array}$$

commute. But by the universal property of the pair $(\mathsf{R}'F, \varepsilon'_F)$, such a natural transformation is unique. It is clear that the identity transformation $\mathrm{id}_{\mathsf{R}'F\circ Q_{\mathcal{A}}}$ also makes this diagram commute and so $\eta \circ \eta' = \mathrm{id}_{\mathsf{R}'F\circ Q_{\mathcal{A}}}$. Similarly, $\eta' \circ \eta = \mathrm{id}_{\mathsf{R}F\circ Q_{\mathcal{A}}}$ and so η and η' are mutually inverse natural isomorphisms.

<u>Construction of $\mathsf{R}F$:</u> Let

$$\Phi : (\mathsf{Qis}^+_{\mathfrak{R}})^{-1}\mathsf{K}^+(\mathfrak{R}) \to \mathsf{D}^+(\mathcal{A})$$

be the equivalence of categories provided by Corollary 5.2.6. Then we can always find a functor

$$\Psi : \mathsf{D}^+(\mathcal{A}) \to (\mathsf{Qis}^+_{\mathfrak{R}})^{-1}\mathsf{K}^+(\mathfrak{R})$$

together with natural isomorphisms

$$\begin{aligned} \alpha &: \mathrm{id}_{(\mathsf{Qis}^+_{\mathfrak{R}})^{-1}\mathsf{K}^+(\mathfrak{R})} \to \Phi \circ \Psi \\ \beta &: \mathrm{id}_{\mathsf{D}^+(\mathcal{A})} \to \Psi \circ \Phi \end{aligned}$$

Let $\overline{F} : (\mathsf{Qis}^+_{\mathfrak{R}})^{-1}\mathsf{K}^+(\mathfrak{R}) \to \mathsf{K}^+(\mathcal{B})$ be the functor given by applying F component-wise. We claim that $\overline{F}(\mathsf{Qis}^+_{\mathfrak{R}}) \subseteq \mathsf{Qis}^+_{\mathcal{B}}$ so that $\overline{F}$ induces a functor

$$\overline{F} : (\mathsf{Qis}^+_{\mathfrak{R}})^{-1}\mathsf{K}^+(\mathfrak{R}) \to \mathsf{D}^+(\mathcal{B})$$

Since $\mathfrak{R}$ is adapted to F, $\overline{F}$ sends acyclic complexes to acyclic complexes. Recall that a morphism is a quasi-isomorphism if and only if its mapping cone is acyclic. Hence

$$\begin{aligned} f \in \mathsf{Qis}_{\mathfrak{R}} &\iff C(f) \text{ is acyclic} \\ &\implies \overline{F}(C(f)) \text{ is acyclic} \\ &\iff C(\overline{F}(f)) \text{ is acyclic} \qquad (\overline{F}(C(f)) \cong C(\overline{F}(f))) \\ &\iff \overline{F}(f) \in \mathsf{Qis}_{\mathcal{B}} \end{aligned}$$

Note that this functor is the unique functor making the diagram

$$\begin{array}{ccc} \mathsf{K}^+(\mathfrak{R}) & \xrightarrow{F^+} & \mathsf{K}^+(\mathcal{B}) \\ \downarrow{\scriptstyle Q_{\mathfrak{R}}} & & \downarrow{\scriptstyle Q_{\mathcal{B}}} \\ (\mathsf{Qis}^+_{\mathfrak{R}})^{-1}\mathsf{K}^+(\mathfrak{R}) & \xrightarrow{\overline{F}} & \mathsf{D}^+(\mathcal{B}) \end{array}$$

commute. Moreover, it is immediate that $\overline{F}$ is a δ-functor. We then define $\mathsf{R}F$ to be the composition

$$\mathsf{R}F = \overline{F} \circ \Phi : \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$$

<u>$\mathsf{R}F$ is a δ-functor:</u> This follows immediately from Lemma 5.3.3.

<u>Construction of the natural transformation ε_F:</u> Fix a complex $X^\bullet \in \mathrm{ob}\,\mathsf{Com}^+(\mathcal{A})$. We need to construct a family of morphisms $(\varepsilon_F)_{X^\bullet} : (Q_{\mathcal{B}} \circ F^+)(X^\bullet) \to (\mathsf{R}F \circ Q_{\mathcal{A}})(X^\bullet)$ such that for every morphism of complexes $f \in \mathsf{Com}^+(\mathcal{A})(X_1^\bullet, X_2^\bullet)$ we have a commutative diagram

$$\begin{array}{ccc} (Q_{\mathcal{B}} \circ F^+)(X_1^\bullet) & \xrightarrow{(Q_{\mathcal{B}}\circ F^+)(f)} & (Q_{\mathcal{B}} \circ F^+)(X_2^\bullet) \\ \downarrow{\scriptstyle (\varepsilon_F)_{X_1^\bullet}} & & \downarrow{\scriptstyle (\varepsilon_F)_{X_2^\bullet}} \\ (\mathsf{R}F \circ Q_{\mathcal{A}})(X_1^\bullet) & \xrightarrow{(\mathsf{R}F\circ Q_{\mathcal{A}})(f)} & (\mathsf{R}F \circ Q_{\mathcal{A}})(X_2^\bullet) \end{array}$$

To this end, choose a right $\mathfrak{R}$-resolution of $X^\bullet$, say $X'^\bullet$ such that $X'^\bullet = (\Phi \circ Q_{\mathcal{A}})(X^\bullet)$. Note that we have an isomorphism of complexes

$$\beta_{X^\bullet} : X^\bullet \to (\Psi \circ \Phi)(X^\bullet) = \Psi(X'^\bullet)$$

in $\mathsf{D}^+(\mathcal{A})$. Then $\beta_{X^\bullet}$ is represented by a diagram $X^\bullet \xrightarrow{s} Z^\bullet \xleftarrow{t} X'$ with $s, t \in \mathsf{Qis}^+_{\mathcal{A}}$. By replacing $Z^\bullet$ with an right $\mathfrak{R}$-resolution of $Z^\bullet$ if necessary, we may assume that $Z^\bullet \in \operatorname{ob} \mathsf{Com}^+(\mathfrak{R})$. Applying the functor $F^+ : \mathsf{K}^+(\mathcal{A}) \to \mathsf{K}^+(\mathcal{B})$ we get a diagram

$$F^+(X^\bullet) \xrightarrow{F^+(s)} F^+(Z^\bullet) \xleftarrow{F^+(t)} F^+(X'^\bullet)$$

Since $\mathfrak{R}$ is adapted to F, $F^+(t)$ is a quasi-isomorphism. Hence applying $Q_{\mathcal{B}}$ we have a morphism

$$(Q_{\mathcal{B}} \circ F^+)(X^\bullet) \xrightarrow{(Q_{\mathcal{B}} \circ F^+)(s)} (Q_{\mathcal{B}} \circ F^+)(Z^\bullet) \xrightarrow{(Q_{\mathcal{B}} \circ F^+)(t)^{-1}} (Q_{\mathcal{B}} \circ F^+)(X'^\bullet)$$

This then defines the desired morphism

$$\begin{aligned}(\varepsilon_F)_{X^\bullet} : (Q_{\mathcal{B}} \circ F^+)(X^\bullet) \to (Q_{\mathcal{B}} \circ F)(X'^\bullet) &= (\overline{F} \circ Q_{\mathfrak{R}})(X'^\bullet) \\ &= (\overline{F} \circ \Phi \circ Q_{\mathcal{A}})(X^\bullet) \\ &= (\mathsf{R}F \circ Q_{\mathcal{A}})(X)\end{aligned}$$

We must, however, check that $(\varepsilon_F)_{X^\bullet}$ is independent of the choice of representative of $\beta_{X^\bullet}$. To this end, suppose that $X^\bullet \xrightarrow{s_1} Z_1^\bullet \xleftarrow{t_1} X'^\bullet$ and $X^\bullet \xrightarrow{s_2} Z_2^\bullet \xleftarrow{t_2} X'^\bullet$ are two representatives of $\beta_{X^\bullet}$. Then we can find $Z_3^\bullet \in \mathsf{K}^+(\mathfrak{R})$ together with morphisms $x : Z_3^\bullet \to X^\bullet, y : Z_3^\bullet \to X^\bullet$ and $z : Z_3^\bullet \to Z_2^\bullet$ such that the diagram

$$\begin{array}{ccccc} & & Z_1^\bullet & & \\ & \swarrow^{s_1^{-1}} & \uparrow y & \searrow^{t_1^{-1}} & \\ X^\bullet & \xleftarrow{x} & Z_3^\bullet & & X'^\bullet \\ & \nwarrow_{s_2^{-1}} & \downarrow z & \nearrow_{t_2^{-1}} & \\ & & Z_2^\bullet & & \end{array}$$

commutes. It is immediate from the fact that $x, s_1^{-1}, s_2^{-1} \in \mathsf{Qis}^+_{\mathcal{A}}$ that y and z are also quasi-isomorphisms. Moreover, symmetry implies that we can assume that $y, z \in \mathsf{Qis}^+_{\mathfrak{R}}$. Relabelling inverses, we thus have morphisms $y : Z_3^\bullet \to Z_1^\bullet$ and $z : Z_3^\bullet \to Z_2^\bullet$ such that $x = y \circ s_1 = z \circ s_2$ and $y \circ t_1 = z \circ t_2$. It is then immediate that the natural transformation arising from applying the construction above to the pair $(x, y \circ t_1)$ agrees with the ones arising from the pairs (s_1, t_1) and (s_2, t_2).

Finally, we must verify that the family $\{ (\varepsilon_F)_{X^\bullet} \mid X^\bullet \in \operatorname{ob} \mathsf{Com}^+(\mathcal{A}) \}$ give rise to a natural transformation $\varepsilon_F : Q_{\mathcal{B}} \circ F^+ \to \mathsf{R}F \circ Q_{\mathcal{A}}$. Fix a morphism $f : X_1^\bullet \to X_2^\bullet$ in $\mathsf{K}^+(\mathcal{A})$ and denote $Y_1^\bullet = (\Phi \circ Q_{\mathcal{A}})(X_1^\bullet)$ and $Y_2^\bullet = (\Phi \circ Q_{\mathcal{A}})(X_2^\bullet)$. Since β is a natural transformation, we have the commutative diagram

$$\begin{array}{ccc} X_1^\bullet & \xrightarrow{f} & X_2^\bullet \\ \downarrow \beta_{X_1^\bullet} & & \downarrow \beta_{X_2^\bullet} \\ \Psi(Y_1^\bullet) & \xrightarrow{(\Psi\circ\Phi)(f)} & \Psi(Y_2^\bullet) \end{array}$$

We can represent each of the morphisms $\beta_{X_1^\bullet}, \beta_{X_2^\bullet}$ and $(\Psi \circ \Phi)(f)$ by roofs in $\mathsf{K}^+(\mathcal{A})$. Replacing all objects in these roofs with right $\mathfrak{R}$-resolutions if necessary, we may assume that these roofs are in $\mathsf{K}^+(\mathfrak{R})$. Note also that $Y_1^\bullet, Y_2^\bullet \in \operatorname{ob} \mathsf{K}^+(\mathfrak{R})$. The fact that $\mathfrak{R}$ is adapted to F means that applying the functor $F^+ : \mathsf{K}^+(\mathcal{A}) \to \mathsf{K}^+(\mathcal{B})$ to these roofs yields roofs in $\mathsf{K}^+(\mathcal{B})$. These roofs then represent morphisms in $D^+(\mathcal{B})$ so applying the localisation functor yields the desired commutative diagram indicating that ε_F is a natural transformation.

<u>The Universal Property of $(\mathsf{R}F, \varepsilon_F)$:</u> Fix a δ-functor $G : \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$ together with a natural transformation $\varepsilon : Q_\mathcal{B} \circ F^+ \to G \circ Q_\mathcal{A}$. We need to exhibit a natural transformation $\eta : \mathsf{R}F \to G$ such that the diagram

$$\begin{array}{ccc} & Q_\mathcal{B} \circ F^+ & \\ {}^{\varepsilon_F}\swarrow & & \searrow^{\varepsilon} \\ \mathsf{R}F \circ Q_\mathcal{A} & \xrightarrow{\eta_{Q_\mathcal{A}}} & G \circ Q_\mathcal{A} \end{array}$$

commutes. That is to say, given each $X^\bullet \in \operatorname{ob} \mathsf{D}^+(\mathcal{A}) = \operatorname{ob} \mathsf{K}^+(\mathcal{A})$ we need to provide a morphism of complexes $\eta_X^\bullet : \mathsf{R}FX^\bullet \to GX^\bullet$ such that for every morphism $f : X^\bullet \to Y^\bullet$ we have a commutative diagram

$$\begin{array}{ccc} \mathsf{R}FX^\bullet & \xrightarrow{\mathsf{R}F(f)} & \mathsf{R}FY^\bullet \\ \downarrow{\scriptstyle \eta_{X^\bullet}} & & \downarrow{\scriptstyle \eta_{Y^\bullet}} \\ GX^\bullet & \xrightarrow{G(f)} & GY^\bullet \end{array}$$

To this end, fix $X^\bullet \in \operatorname{ob} \mathsf{D}^+(\mathcal{A})$. The natural transformation ε provides a morphism $\varepsilon_X : FX \to GX$ in $\mathsf{D}^+(\mathcal{B})$ for each $X \in \operatorname{ob} \mathsf{K}^+(\mathcal{A})$. Moreover, the natural transformation $\beta : \mathrm{id}_{\mathsf{D}^+(\mathcal{A})} \to \Psi \circ \Phi$ provides an isomorphism $\beta_{X^\bullet} : X^\bullet \to (\Psi \circ \Phi)(X^\bullet)$ in $\mathsf{D}^+(\mathcal{A})$. As before, this isomorphism admits a representative $X^\bullet \xrightarrow{s} Z^\bullet \xleftarrow{t} X'^\bullet$ with $Z^\bullet \in \operatorname{ob} \mathsf{K}^+(\mathfrak{R})$ and $s, t \in \mathsf{Qis}^+_\mathfrak{R}$. By naturality of ε we thus have a commutative diagram

$$\begin{array}{ccccc} F(X^\bullet) & \xrightarrow{(Q_\mathcal{B} \circ F^+)(s)} & F(Z^\bullet) & \xleftarrow{(Q_\mathcal{B} \circ F^+)(t)} & F(X'^\bullet) \\ \downarrow{\scriptstyle \varepsilon_{X^\bullet}} & & \downarrow{\scriptstyle \varepsilon_{Z^\bullet}} & & \downarrow{\scriptstyle \varepsilon_{X'^\bullet}} \\ GX^\bullet & \xrightarrow{(G \circ Q_\mathcal{A})(s)} & GZ^\bullet & \xleftarrow{(G \circ Q_\mathcal{A})(t)} & GX'^\bullet \end{array}$$

in $\mathsf{D}^+(\mathcal{A})$. Since t is a quasi-isomorphism and $\mathfrak{R}$ is adapted for F it follows that $G \circ Q_\mathcal{A}(t)$ and $Q_\mathcal{B} \circ F^+(t)$ are isomorphisms. Inverting these morphisms in the diagram above yields the diagram

$$\begin{array}{ccc} (Q_\mathcal{B} \circ F^+)(X^\bullet) & \xrightarrow{(\varepsilon_F)_{X^\bullet}} & (\mathsf{R}F \circ Q_\mathcal{A})(X^\bullet) \\ \downarrow{\scriptstyle \varepsilon_{X^\bullet}} & & \downarrow{\scriptstyle \varepsilon_{(\Psi \circ \Phi)(X^\bullet)}} \\ GX^\bullet & \xrightarrow{G\beta_{X^\bullet}} & G((\Psi \circ \Phi)(X^\bullet)) \end{array} \tag{1}$$

Since $\beta_{X^\bullet}$ is an isomorphism, so is $G\beta_{X^\bullet}$. We now claim that

$$\eta_{X^\bullet} = (G\beta_{X^\bullet})^{-1} \circ \varepsilon_{(\Psi \circ \Phi)(X^\bullet)} : \mathsf{R}FX^\bullet \to GX^\bullet$$

defines the desired unique natural transformation. Firstly, it is clear that we obtain the desired commutative diagram of natural transformations by the definition of η. To check that η is indeed a natural transformation, let $\lambda : X^\bullet \to Y^\bullet$ be a morphism of complexes in $\mathsf{D}^+(\mathcal{A})$. We need to check that the diagram

$$\begin{array}{ccc}
\mathsf{R}FX^\bullet & \xrightarrow{RF(\lambda)} & \mathsf{R}FY^\bullet \\
\downarrow{\scriptstyle \eta_{X^\bullet}} & & \downarrow{\scriptstyle \eta_{Y^\bullet}} \\
GX^\bullet & \xrightarrow{G(\lambda)} & GY^\bullet
\end{array}$$

commutes. Note that, since β is a natural isomorphism, we have a commutative diagram

$$\begin{array}{ccc}
GX^\bullet & \xrightarrow{\qquad G(\lambda) \qquad} & GY^\bullet \\
{\scriptstyle (G\beta_{X^\bullet})^{-1}}\uparrow & & {\scriptstyle (G\beta_{Y^\bullet})^{-1}}\uparrow \\
G((\Psi \circ \Phi)(X^\bullet)) & \xrightarrow{G((\Psi\circ\Phi)(\lambda))} & G((\Psi \circ \Phi)(Y^\bullet))
\end{array}$$

This implies that

$$G(\lambda) \circ \eta_{X^\bullet} = G(\lambda) \circ (G\beta_{X^\bullet})^{-1} \circ \varepsilon_{(\Psi\circ\Phi)(X^\bullet)} = (G\beta_{Y^\bullet})^{-1} \circ G((\Psi \circ \Phi)(\lambda)) \circ \varepsilon_{(\Psi\circ\Phi)(X^\bullet)}$$

Now suppose that λ admits a representative roof $s^{-1}f : X^\bullet \to Z^\bullet \to Y^\bullet$. Then the naturality of ε provides us with a commutative diagram

$$\begin{array}{ccccc}
(\mathsf{R}F \circ Q_{\mathcal{A}})(X^\bullet) & \xleftarrow{(\mathsf{R}F\circ Q_{\mathcal{A}})(s)} & (\mathsf{R}F \circ Q_{\mathcal{A}})(Z^\bullet) & \xrightarrow{(\mathsf{R}F\circ Q_{\mathcal{A}})(f)} & (\mathsf{R}F \circ Q_{\mathcal{A}})(Y^\bullet) \\
\downarrow{\scriptstyle \varepsilon_{(\Psi\circ\Phi)(X^\bullet)}} & & \downarrow{\scriptstyle \varepsilon_{(\Psi\circ\Phi)(Z^\bullet)}} & & \downarrow{\scriptstyle \varepsilon_{(\Psi\circ\Phi)(Y^\bullet)}} \\
G((\Psi \circ \Phi)(X^\bullet)) & \xleftarrow{G((\Psi\circ\Phi)(s))} & G((\Psi \circ \Phi)(Z^\bullet)) & \xrightarrow{G((\Psi\circ\Phi)(f))} & G((\Psi \circ \Phi)(Y^\bullet))
\end{array}$$

Since s is a quasi-isomorphism, it follows that that the two left-hand horiztonal arrows are invertible and so we get an induced diagram

$$\begin{array}{ccc}
(\mathsf{R}F \circ Q_{\mathcal{A}})(X^\bullet) & \xrightarrow{\qquad \mathsf{R}F(\lambda) \qquad} & (\mathsf{R}F \circ Q_{\mathcal{A}})(Y^\bullet) \\
\downarrow{\scriptstyle \varepsilon_{(\Psi\circ\Phi)(X^\bullet)}} & & \downarrow{\scriptstyle \varepsilon_{(\Psi\circ\Phi)(Y^\bullet)}} \\
G((\Psi \circ \Phi)(X^\bullet)) & \xrightarrow{G((\Psi\circ\Phi)(s))} & G((\Psi \circ \Phi)(Y^\bullet))
\end{array}$$

in $\mathsf{D}^+(\mathcal{B})$. Hence

$$\begin{aligned}
G(\lambda) \circ \eta_{X^\bullet} &= (G\beta_{Y^\bullet})^{-1} \circ G((\Psi \circ \Phi)(\lambda)) \circ \varepsilon_{(\Psi\circ\Phi)(X^\bullet)} \\
&= (G\beta_{Y^\bullet})^{-1} \circ \varepsilon_{(\Psi\circ\Phi)(Y^\bullet)} \circ \mathsf{R}F(\lambda) \\
&= \eta_{Y^\bullet} \circ \mathsf{R}F(\lambda)
\end{aligned}$$

so that η is indeed a natural transformation. It is evident from the construction of η that it is the unique such natural transformation. □

Definition 5.3.5. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor and suppose that $\mathsf{R}F$ exists. We define the **higher** (or **classical**) **derived functors** of F to be $\mathsf{R}^i F = H^i(\mathsf{R}F)$ for all $i \in \mathbb{Z}$.

Proposition 5.3.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor and suppose that $\mathcal{A}$ has enough right $\mathfrak{R}$-objects. Then*

1. $\mathsf{R}^i F = 0$ *for all* $i < 0$.
2. $\mathsf{R}^0 F = F$.

3. Given a short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in $\mathcal{A}$, there is a corresponding long-exact sequence of higher right-derived functors

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathsf{R}^0FX & \longrightarrow & \mathsf{R}^0FY & \longrightarrow & \mathsf{R}^0FZ \\ & \longrightarrow & \mathsf{R}^1FX & \longrightarrow & \mathsf{R}^1FY & \longrightarrow & \mathsf{R}^1FZ \\ & \dashrightarrow & \mathsf{R}^nFX & \longrightarrow & \mathsf{R}^nFY & \longrightarrow & \mathsf{R}^nFZ \end{array}$$

Proof. Fix an object $X \in \operatorname{ob}\mathcal{A}$ and choose a right $\mathfrak{R}$ resolution $f : A[0] \to R^\bullet$ of A. Then $\mathsf{R}^iF(A) = H^i(F(R^\bullet))$. It is then clear that $\mathsf{R}^iF = 0$ for $i < 0$. Moreover, since f is a quasi-isomorphism, we have an isomorphism $A \cong \ker(R^0 \to R^1)$. Since F is left-exact, we then have an isomorphism $FA \cong \ker(FR^0 \to FR^1)$. Taking H^0 of both sides of this isomorphism, we see that $FA \cong \mathsf{R}^0FA$. That this isomorphism is natural in A follows immediately from the definitions.

Finally, given the short exact sequence in the statement, Proposition 4.6.3 implies that there exists a triangle

$$X[0] \longrightarrow Y[0] \longrightarrow Z[0] \longrightarrow X[1]$$

in $\mathsf{D}^+(\mathcal{A})$. Since $\mathsf{R}F$ is a δ-functor, we then have a triangle

$$\mathsf{R}FX[0] \longrightarrow \mathsf{R}FY[0] \longrightarrow \mathsf{R}FZ[0] \longrightarrow \mathsf{R}FX[1]$$

in $\mathsf{D}^+(\mathcal{B})$. Appealing to the long-exact sequence of cohomology, together with Part 1 and 2, yields the desired long-exact sequence of higher right derived functors. □

Proposition 5.3.7. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}, G : \mathcal{B} \to \mathcal{C}$ left-exact functors. Suppose that $\mathfrak{R}$ and $\mathfrak{R}'$ are collections of objects in $\mathcal{A}$ and $\mathcal{B}$ respectively that are adapted to F and G respectively. Assume, moreover, that $F(\mathfrak{R}) \subseteq \mathfrak{R}'$. Then the right derived functor $\mathsf{R}(G \circ F)$ exists and the natural morphism of functors*

$$\mathsf{R}(G \circ F) \to \mathsf{R}G \circ \mathsf{R}F$$

is an isomorphism.

Proof. Since $F(\mathfrak{R}) \subseteq \mathfrak{R}'$, it follows that $\mathfrak{R}$ is adapted to $G \circ F$. Hence the right derived functor $\mathsf{R}(G \circ F)$ exists. Note that since $\mathsf{R}F$ and $\mathsf{R}G$ are δ-functors, so is $\mathsf{R}G \circ \mathsf{R}G$. $\mathsf{R}(G \circ F)$ is moreover a δ-functor and so the universal property of derived functors provides us with a natural transformation

$$\mathsf{R}(G \circ F) \to \mathsf{R}G \circ \mathsf{R}F$$

Now for $R^\bullet \in \mathsf{K}^+(\mathfrak{R})$, this natural transformation is clearly an isomorphism. For the general case, we can reduce to this case by choosing a right $\mathfrak{R}$-resolution $X^\bullet \to R^\bullet$ for each $X \in \mathsf{D}^+(\mathcal{A})$ which is an isomorphism in $\mathsf{D}^+(\mathcal{A})$. □

Definition 5.3.8. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor and suppose that $\mathsf{R}F$ exists. We say that $X \in \operatorname{ob}\mathcal{A}$ is ***F*-acyclic** if $\mathsf{R}^iFX = 0$ for all $i \neq 0$.

Proposition 5.3.9. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor such that $\mathsf{R}F$ exists and denote by $\mathfrak{Z}$ the collection of all F-acyclic objects of $\mathcal{A}$.*

1. *A collection of objects that are adapted to F exists if and only if $\mathcal{A}$ has enough right $\mathfrak{Z}$-objects.*

2. *If $\mathcal{A}$ has enough right $\mathfrak{Z}$-objects then $\mathfrak{Z}$ contains any collection of objects adapted to F and any subcollection $\mathfrak{Z}'$ of $\mathfrak{Z}$ such that $\mathcal{A}$ has enough right $\mathfrak{Z}'$-objects is adapted to F.*

Proof. First suppose that there exists a collection $\mathfrak{R}$ of objects of $\mathcal{A}$ that is adapted to F. Fix $X \in \mathfrak{R}$. Then by the definition of $\mathsf{R}F$, we have that $\mathsf{R}F(X[0])$ is quasi-isomorphic to $(FX)[0]$. Hence $X \in \mathfrak{Z}$ and so $\mathcal{A}$ has enough right $\mathfrak{Z}$-objects.

Now note that the converse of Part 1 and Part 2 will follow if we can prove that, given a subcollection $\mathfrak{R} \subseteq \mathfrak{Z}$ such that $\mathcal{A}$ has enough right $\mathfrak{R}$-objects, F maps acyclic complexes in $\mathsf{Com}^+(\mathfrak{R})$ to acyclic complexes.

First suppose that we are given an acylic triple $X^\bullet$ represented by $0 \to X^0 \to X^1 \to X^2 \to 0$. Then this sequence is a short-exact sequence. Applying F yields a left-exact sequence

$$0 \longrightarrow FX^0 \longrightarrow FX^1 \longrightarrow FX^2 \longrightarrow 0$$

Since $X^\bullet \in \operatorname{ob}\mathsf{Com}^+(\mathfrak{R}) \operatorname{ob}\mathsf{Com}^+(\mathfrak{Z})$, it follows that $\mathsf{R}^1FX^0 = 0$ and so we also have exactness from the right. Hence $F(X^\bullet)$ is also ayclic.

Now let $X^\bullet \in \operatorname{ob}\mathsf{Com}^+(\mathfrak{R})$ be a general acyclic complex. We can reduce to the case of short exact sequences as follows. Set $Y^0 = X^0$ and $Y^i = \operatorname{im} d_X^i$. Then we get short-exact sequences

$$0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow Y^1 \longrightarrow 0$$

$$0 \longrightarrow Y^1 \longrightarrow X^2 \longrightarrow Y^2 \longrightarrow 0$$

and so on with each $Y^i \in \operatorname{ob}\mathfrak{Z}$. It then follows that each sequence

$$0 \longrightarrow FY^i \longrightarrow FX^{i+1} \longrightarrow FY^{i+1} \longrightarrow 0$$

is exact so that $FX^\bullet$ is acyclic. $\square$

5.4 Injective Resolutions

Throughout this section, let $\mathcal{A}$ be an abelian category and $\mathfrak{I}$ be the collection of injective objects in $\mathcal{A}$.

Lemma 5.4.1. *Let $X^\bullet \in \mathsf{K}^+(\mathcal{A})$ be an acyclic complex. If $I^\bullet \in \mathsf{K}^+(\mathfrak{I})$ then $\mathsf{K}(\mathcal{A})(X^\bullet, I^\bullet) = 0$*

Proof. Fix a morphism $u \in \mathsf{Com}(\mathcal{A})(X^\bullet, I^\bullet)$. We need to show that u is null-homotopic. That is to say, we need to exhibit a morphism $k \in \mathcal{A}^{-1}(X^\bullet, I^\bullet)$ such that $d_I k + k d_X = u$. Without loss of generality, we may assume that $I^i = 0$ for all $i < 0$. If such a k were to exist then it is evident that $k^i = 0$ for $i \leq 0$. We shall construct k by induction on i.

Fix $y \in \operatorname{im}(d_X^j : X^q \to X^{j+1})$. Let $x \in X^j$ be such that $d_X^j(x) = y$. Given any other $x' \in X^j$ such that $d_X^j(x') = y$, we necessarily have that there exists $w \in X^{j-1}$ such that $d_X^{j-1}(w) = x - x'$. By hypothesis, we then have that

$$\begin{aligned} u^j(x) - (d_I^{j-1} \circ k^j)(x) = (k^{j+1}(x) \circ d_X^j)(x) = k^{j+1}(y) &= (k^{j+1} \circ d_X^j)(x') \\ &= u^j(x') - (d_I^{j-1} \circ k^j)(x') \end{aligned}$$

This then implies that

$$\begin{aligned} u^j(x-x') - (d_I^{j-1}\circ k^j)(x-x') &= (u^j \circ d_X^{j-1}(w) - (d_I^{j-1}\circ k^j \circ d_X^{j-1}(w)) \\ &= (d_I^{j-1}\circ u^{j-1})(w) - d_I^{j-1}(u^{j-1}(w) + d_I^{j-2}\circ k^{j+1}(w)) \\ &= 0 \end{aligned}$$

Now define a map $k^{j+1} : \operatorname{im} d_X^j \to I^q$ by setting

$$k^{j+1}(y) = u^j(x) - (d_I^{j-1}\circ k^j)(x)$$

Then the above argumentation ensures that k^{j+1} is independent of the choice of pre-image of y. Since I^{j+1} is injective, there exists a morphism $\overline{k^{j+1}}$ such that the diagram

$$\begin{array}{ccc} & & X^{j+1} \\ & \nearrow & \downarrow{\scriptstyle \overline{k^{j+1}}} \\ \operatorname{im} d_X^j & \xrightarrow{k^j} & I^{j+1} \end{array}$$

commutes. Since $X^\bullet$ is acyclic, $\operatorname{im} d_X^j = \ker d_X^j$. Hence for $y \in X^{q+1}$ we have

$$(\overline{k^{j+1}}\circ d_X^j)(y) = (k^{j+1}\circ d_X^j)(y) = u^j(y) - (d_I^{j-1}\circ k^j)(y)$$

which ensures that $\overline{k^{j+1}}$ extends the map k^j into a homotopy operator up to $i = j+1$. □

Proposition 5.4.2. *Let $s \in \mathsf{Com}^+(\mathfrak{I})(X^\bullet, Y^\bullet)$ be a quasi-isomorphism. Then s is null-homotopic.*

Proof. Consider the standard triangle

$$X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{q} C(u)^\bullet \longrightarrow X[1]^\bullet$$

in $\mathsf{K}^+(\mathfrak{I})$. Since u is a quasi-isomorphism, $C(u)$ is acyclic. By Lemma 5.4.1, p is thus null-homotopic. More precisely, there exists $k \in \mathcal{A}^{-1}(C(u)^\bullet, X[1]^\bullet)$ such that $d_Y k + k d_{X[1]} = p$. Right-composing this equation with q and noting that k has degree -1, we see that $-d_Y kq + k d_{X[1]} q = 0$. Since q is a morphism of complexes, it then follows that $kqd_{X[1]} = d_Y kq$ so that $v = kq$ (which has degree 0) is a morphism of complexes.

Now let $q_{X^\bullet}$ be the $X^\bullet$ component of q. Then $k' = kq_{X^\bullet}$ is a degree -1 morphism. By the definition of the differential $d_{C(u)^\bullet}$, we have

$$d_X k' = k' d_X = vu - \mathrm{id}_{X^\bullet}$$

and so $vu \sim \mathrm{id}_{X^\bullet}$. Moreover, we have that $v^* u^* = \mathrm{id}_{H(X^\bullet)}$ so that v is a quasi-isomorphism. Repeating the previous argumentation to v yields a morphism $w \in C^+(\mathfrak{I})(X^\bullet, Y^\bullet)$ such that $wv \sim 1_{Y^\bullet}$. We then see that $w \sim u$ which implies that $1_{Y^\bullet} \sim uv$ whence u is null-homotopic. □

Corollary 5.4.3. *There is an equivalence of categories $(\mathsf{Qis}_{\mathfrak{I}}^+)^{-1}\mathsf{K}^+(\mathfrak{I}) \cong \mathsf{K}^+(\mathfrak{I})$. If, moreover, $\mathcal{A}$ has enough injectives then we have an equivalence of categories $\mathsf{K}^+(\mathfrak{I}) \cong \mathsf{D}^+(\mathcal{A})$.*

Proof. By Proposition 5.4.2, every quasi-isomorphism is null-homotopic. Hence the localisation functor

$$\mathsf{K}^+(\mathfrak{I}) \to (\mathsf{Qis}_{\mathfrak{I}}^+)^{-1}\mathsf{K}^+(\mathfrak{I})$$

is necessarily an equivalence. Moreover, if $\mathcal{A}$ has enough injectives then we have a chain of equivalences

$$\mathsf{K}^+(\mathfrak{I}) \cong (\mathsf{Qis}_{\mathfrak{I}}^+)^{-1}\mathsf{K}^+(\mathfrak{I}) \cong \mathsf{D}^+(\mathcal{A})$$

by Proposition 5.2.6. □

Corollary 5.4.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor and suppose that $\mathcal{A}$ has enough injectives. Then the collection $\mathfrak{I}$ of injective objects in $\mathcal{A}$ is adapted to F.*

Proof. We need to show that F maps acyclic complexes in $\mathsf{Com}^+(\mathfrak{I})$ to acyclic complexes. To this end, fix an acyclic complex $I^\bullet \in \mathsf{Com}^+(\mathfrak{I})$. Then the zero morphism $0 : I^\bullet \to I^\bullet$ is necessarily a quasi-isomorphism. By Proposition 5.4.2, it is homotopic to $\mathrm{id}_{I^\bullet}$. Hence the zero morphism of $F(I^\bullet)$ is homotopic to $\mathrm{id}_{F^\bullet}$. It follows that $F(I^\bullet)$ is acyclic as desired. □

6 Inner Hom

Throughout this section, $\mathcal{A}$ will be an abelian category.

6.1 Definition and Properties

Definition 6.1.1. Let $X^\bullet, Y^\bullet \in \mathrm{ob}\,\mathsf{Com}(\mathcal{A})$. Then we define the **inner hom** of $X^\bullet$ and $Y^\bullet$ to be the complex $\mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, Y^\bullet) \in \mathsf{Com}(\mathsf{AbGrp})$ given by the data

$$\mathrm{Hom}^i_{\mathcal{A}}(X^\bullet, Y^\bullet) = \prod_{p\in\mathbb{Z}} \mathsf{Com}(\mathcal{A})(X^p, Y[i]^p)$$
$$d^i_{X^\bullet\to Y^\bullet}(\phi) = (d_Y^{p+i} \circ \phi^p - (-1)^i(\phi^{p+1} \circ d_X^p))_{p\in\mathbb{Z}}$$

for $\phi \in \mathrm{Hom}^i_{\mathcal{A}}(X^\bullet, Y^\bullet)$ where we understand ϕ^p to be the p^{th}-component of the morphism ϕ. If it is clear from context which category we are discussing, we shall simply write $\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet) := \mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, Y^\bullet)$.

Given morphisms $u \in \mathsf{Com}(\mathcal{A})(X'^\bullet, X^\bullet)$ and $v \in \mathsf{Com}(\mathcal{A})(Y^\bullet, Y'^\bullet)$, we define a morphism of complexes

$$\mathrm{Hom}^\bullet(u, v) : \mathrm{Hom}^\bullet(X^\bullet, Y^\bullet) \to \mathrm{Hom}^\bullet(X'^\bullet, Y'^\bullet)$$

by setting for each $i \in \mathbb{Z}$ and $\phi = (\phi^p)_{p\in\mathbb{Z}} \in \mathrm{Hom}^i(X^\bullet, Y^\bullet)$

$$\mathrm{Hom}^i(u, v)(\phi) = ((-1)^i v^{i+p} \circ \phi^p \circ u^p)_{p\in\mathbb{Z}}$$

This then defines the bifunctor **inner hom**

$$\mathrm{Hom}^\bullet(-,-) : \mathsf{Com}(\mathcal{A})^{\mathrm{op}} \times \mathsf{Com}(\mathcal{A}) \to \mathsf{Com}(\mathsf{AbGrp})$$

Lemma 6.1.2. *Let $X^\bullet, Y^\bullet \in \mathrm{ob}\, Com(\mathcal{A})$ be complexes.*

1. *The i^{th} cocyles of $\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)$ are precisely $\mathsf{Com}(\mathcal{A})(X^\bullet, Y[i]^\bullet)$.*
2. *The i^{th} coboundaries of $\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)$ are precisely the null-homotopic morphisms in $\mathsf{Com}(\mathcal{A})(X^\bullet, Y[i]^\bullet)$.*
3. $H^i(\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)) = \mathsf{K}(\mathcal{A})(X^\bullet, Y[i]^\bullet)$

Proof.

Part 1: We have that

$$\begin{aligned}
\phi \in Z^i \operatorname{Hom}^\bullet(X^\bullet, Y^\bullet) &\iff d^i_{X^\bullet \to Y^\bullet}(\phi) = 0 \\
&\iff d_Y^{p+i} \circ \phi^p - (-1)^i(\phi^{p+1} \circ d_X^p) = 0 \qquad (\forall p \in \mathbb{Z}) \\
&\iff d_Y^{p+i} \circ \phi^p = (-1)^i(\phi^{p+1} \circ d_X^p) \qquad (\forall p \in \mathbb{Z}) \\
&\iff \phi \in \mathsf{Com}(\mathcal{A})(X^\bullet, Y[i]^\bullet)
\end{aligned}$$

Part 2: We have that

$$\begin{aligned}
\phi \in B^i \operatorname{Hom}^\bullet(X^\bullet, Y^\bullet) &\iff \phi \in \operatorname{im} d^{i-1}_{X^\bullet \to Y^\bullet} \\
&\iff \text{there exists } \psi \in \operatorname{Hom}^{i-1}(X^\bullet, Y^\bullet) \text{ such that} \\
&\qquad \phi^p = d_Y^{p+i-1} \circ \psi^p - (-1)^i(\psi^{p+1} \circ d_X^p) \qquad (\forall p \in \mathbb{Z}) \\
&\iff \phi \text{ is homotopic to 0 in } \mathsf{Com}(\mathcal{A})(X^\bullet, Y[i]^\bullet)
\end{aligned}$$

Part 3: This follows immediately from Part 1 and 2.

□

Lemma 6.1.3. *Let* $u : X'^\bullet \to X^\bullet$ *and* $v : Y^\bullet \to Y'^\bullet$ *be morphisms of complexes in* $\mathsf{Com}(\mathcal{A})$. *Consider the complex*

$$H^\bullet = \operatorname{Hom}^\bullet_{\mathsf{AbGrp}}(\operatorname{Hom}_{\mathcal{A}}(X^\bullet, Y^\bullet), \operatorname{Hom}_{\mathcal{A}}(X'^\bullet, Y'^\bullet))$$

in $\mathsf{Com}(\mathsf{AbGrp})$. *Then*

$$d_H(\operatorname{Hom}^\bullet_{\mathcal{A}}(u, v)) = \operatorname{Hom}^\bullet_{\mathcal{A}}(d_{X'^\bullet \to X^\bullet}(u), v) + \operatorname{Hom}^\bullet_{\mathcal{A}}(u, d_{Y^\bullet \to Y'^\bullet}(v))$$

Proof. In this proof we omit signs and indices in favour of clarity. On one hand we have that

$$\begin{aligned}
d_H(\operatorname{Hom}^\bullet(u, v))(\phi) &= (d_{X'^\bullet \to Y'^\bullet} \circ \operatorname{Hom}^\bullet(u, v))(\phi) + (\operatorname{Hom}^\bullet(u, v) \circ d_{X^\bullet \to Y^\bullet})(\phi) \\
&= d_{X'^\bullet \to Y'^\bullet}(v \circ \phi \circ u) + (v \circ (d_{X^\bullet \to Y^\bullet}(\phi)) \circ u) \\
&= d_{Y'} \circ v \circ \phi \circ u + v \circ \phi \circ u \circ d_{X'} \\
&\quad + v \circ d_Y \circ \phi \circ u + v \circ \phi \circ d_X \circ u
\end{aligned}$$

On the other hand, $\operatorname{Hom}^\bullet_{\mathcal{A}}(d_{X'^\bullet \to X^\bullet}(u), v)(\phi) + \operatorname{Hom}^\bullet_{\mathcal{A}}(u, d_{Y^\bullet \to Y'^\bullet}(v))(\phi)$ is given by

$$\begin{aligned}
v \circ \phi \circ d_{X'^\bullet \to X^\bullet}(u) + d_{Y^\bullet \to Y'^\bullet}(v) \circ \phi \circ u &= v \circ \phi \circ d_X \circ u \\
&\quad + v \circ \phi \circ u \circ d_{X'} \\
&\quad + d_{Y'} \circ v \circ \phi \circ u \\
&\quad + v \circ d_Y \circ \phi \circ u
\end{aligned}$$

These two expressions evidently coincide whence the Lemma. □

Proposition 6.1.4. $\operatorname{Hom}^\bullet(-,-)$ *is an additive functor that preserves homotopy equivalences and thus descends to an additive a functor*

$$\operatorname{Hom}^\bullet(-,-) : \mathsf{K}(\mathcal{A})^{\mathrm{op}} \times \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathsf{AbGrp})$$

Proof. That $\mathrm{Hom}^\bullet(-,-)$ is additive follows immediately from the definition. Suppose that $v : Y^\bullet \to Y'^\bullet$ is null-homotopic. By Lemma 6.1.2, v is a 0^{th} coboundary in $\mathrm{Hom}^\bullet(Y^\bullet, Y'^\bullet)$. Hence there exists $z \in \mathrm{Hom}^{-1}(Y^\bullet, Y'^\bullet)$ such that $d^{i-1}_{Y^\bullet \to Y'^\bullet}(z) = v$. Then by Lemma 6.1.3 we have

$$\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathrm{id}_{X'^\bullet}, v) = \mathrm{Hom}_{\mathcal{A}}(\mathrm{id}_{X'^\bullet}, d_{Y^\bullet \to Y'^\bullet}(z)) = d_H(\mathrm{Hom}^\bullet_{\mathcal{A}}(u, z)) - \mathrm{Hom}^\bullet_{\mathcal{A}}(d_{X'^\bullet \to X^\bullet}(\mathrm{id}_{X'^\bullet}), z)$$

so that $\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathrm{id}_{X'^\bullet}, v)$ is null-homotopic. A similar argument shows that $\mathrm{Hom}^\bullet(-,-)$ preserves homotopy equivalences in the first argument. □

Lemma 6.1.5. *Let $W^\bullet \in \mathsf{Com}(\mathcal{A})$ be a complex and $u \in \mathsf{Com}(\mathcal{A})(X^\bullet, Y^\bullet)$ a morphism. Denote*

$$\overline{u} = \mathrm{Hom}^\bullet(\mathrm{id}_{W^\bullet}, u) : \mathrm{Hom}^\bullet(W^\bullet, X^\bullet) \to \mathrm{Hom}^\bullet(W^\bullet, Y^\bullet)$$
$$\underline{u} = \mathrm{Hom}^\bullet(u, \mathrm{id}_{W^\bullet}) : \mathrm{Hom}^\bullet(Y^\bullet, W^\bullet) \to \mathrm{Hom}^\bullet(X^\bullet, W^\bullet)$$

Then $C(\overline{u})^\bullet = \mathrm{Hom}^\bullet(W^\bullet, C(u)^\bullet)$ and $(C(\underline{u})^\bullet)^{\mathrm{op}} = \mathrm{Hom}^\bullet(C(u)^\bullet, W^\bullet)$.

Proof. By definition, we have that

$$C(\overline{u})^n = \mathrm{Hom}^{n+1}(W^\bullet, X^\bullet) \oplus \mathrm{Hom}^n(W^\bullet, Y^\bullet)$$

and, on the other hand,

$$\begin{aligned}\mathrm{Hom}^n(W^\bullet, C(u)^\bullet) = \mathrm{Hom}^n(W^\bullet, X[1]^\bullet \oplus Y^\bullet) &= \mathrm{Hom}^n(W^\bullet, X[1]^\bullet) \oplus \mathrm{Hom}^n(W^\bullet, Y^\bullet) \\ &= \mathrm{Hom}^{n+1}(W^\bullet, X^\bullet) \oplus \mathrm{Hom}^n(W^\bullet, Y^\bullet)\end{aligned}$$

We thus have a canonical isomorphism $\pi^n : \mathrm{Hom}^n(W^\bullet, C(u)^\bullet) \to C(\underline{u})^n$ given by

$$\begin{aligned}\pi^n : \mathrm{Hom}^n(W^\bullet, X[1]^\bullet \oplus Y^\bullet) &\to \mathrm{Hom}^{n+1}(W^\bullet, X^\bullet) \oplus \mathrm{Hom}^n(W^\bullet, Y^\bullet) \\ \phi &\mapsto (\pi_1 \circ \phi, \pi_2 \circ \phi)\end{aligned}$$

where π_1 and π_2 are the canonical projections onto the components of $X[1]^\bullet \oplus Y^\bullet$. That π^n is a morphism of complexes follows immediately from the fact that π_1 and π_2 are morphisms of complexes. The argumentation for the first argument of inner hom follows the same principle. □

6.2 Derived Functor of Inner Hom

Proposition 6.2.1. *The functor*

$$\mathrm{Hom}^\bullet(-,-) : \mathsf{K}(\mathcal{A})^{\mathrm{op}} \times \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathsf{AbGrp})$$

is a bi-δ-functor.

Proof. We show that, for fixed $W^\bullet \in \mathrm{ob}\,\mathsf{K}(\mathcal{A})$, the functor $\mathrm{Hom}^\bullet(W^\bullet, -)$ is a δ-functor. The argumentation for the first argument follows the same principle.

First observe that for all $X^\bullet \in \mathsf{K}(\mathcal{A})$ we have

$$\mathrm{Hom}^\bullet(W^\bullet, X[1]^\bullet) = \mathrm{Hom}^\bullet(W^\bullet, X^\bullet)[1]$$

It is then immediate from Lemma 6.1.5 that $\mathrm{Hom}^\bullet(W^\bullet, -)$ transforms distinguished triangles in $\mathsf{K}(\mathcal{A})$ to distinguished triangles in $\mathsf{K}(\mathsf{AbGrp})$. □

Definition 6.2.2. Suppose that $\mathcal{A}$ has enough injectives and let $\mathfrak{I}$ be the collection of injective objects in $\mathcal{A}$. Given an object $X^\bullet \in \operatorname{ob} \mathsf{K}(\mathcal{A})$, the functor $\operatorname{Hom}^\bullet(X^\bullet, -)$ is a δ-functor. By Corollary 5.4.4, $\mathfrak{I}$ is adapted to $\operatorname{Hom}^\bullet(X^\bullet, -)$ (considered as a functor restricted to $\mathsf{K}^+(\mathcal{A})$) so it admits a right derived functor which we denote

$$\mathsf{R}_2 \operatorname{Hom}^\bullet(X^\bullet, -) : \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathsf{AbGrp})$$

to signify we are deriving the functor in the second argument. This induces a bifunctor

$$\mathsf{R}_2 \operatorname{Hom}^\bullet(-, -) : \mathsf{K}(\mathcal{A})^{\mathrm{op}} \times \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathsf{AbGrp})$$

Given $Y^\bullet \in \operatorname{ob} \mathsf{D}^+(\mathcal{A})$, we then get a functor

$$\mathsf{R}_2 \operatorname{Hom}^\bullet(-, Y^\bullet) : \mathsf{K}(\mathcal{A})^{\mathrm{op}} \to \mathsf{D}(\mathsf{AbGrp})$$

Now suppose that $s : X'^\bullet \to X^\bullet$ is a quasi-isomorphism and $I^\bullet$ an injective resolution of $Y^\bullet$. Then $\mathsf{R}_2 \operatorname{Hom}^\bullet(s, Y^\bullet) = \operatorname{Hom}^\bullet(s, I^\bullet)$ and the morphism

$$\operatorname{Hom}^\bullet(s, I^\bullet) : \operatorname{Hom}^\bullet(X^\bullet, I^\bullet) \to \operatorname{Hom}^\bullet(X'^\bullet, I^\bullet)$$

is a quasi-isomorphism since $\mathfrak{I}$ is adapted to $\operatorname{Hom}^\bullet(-, I^\bullet)$ and sends acyclic complexes to acyclic complexes. Hence by the universal property of the localisation functor $Q_{\mathcal{A}}^{\mathrm{op}} : \mathsf{K}(\mathcal{A})^{\mathrm{op}} \to \mathsf{D}(\mathcal{A})^{\mathrm{op}}$, we obtain a unique functor

$$\mathsf{R}_1\mathsf{R}_2 \operatorname{Hom}^\bullet(-, Y^\bullet) : \mathsf{D}(\mathcal{A})^{\mathrm{op}} \to \mathsf{D}(\mathsf{AbGrp})$$

which induces a bifunctor

$$\mathsf{R}_1\mathsf{R}_2 \operatorname{Hom}^\bullet(-, -) : \mathsf{D}(\mathcal{A})^{\mathrm{op}} \times \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathsf{AbGrp})$$

Dually, if $\mathcal{A}$ has enough projectives, we can construct a unique functor

$$\mathsf{R}_2\mathsf{R}_1 \operatorname{Hom}^\bullet(-, -) : \mathsf{D}^-(\mathcal{A})^{\mathrm{op}} \times \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathsf{AbGrp})$$

Proposition 6.2.3. *Suppose that $\mathcal{A}$ has enough injectives and projectives. Then the functors*

$$\mathsf{R}_1\mathsf{R}_2 \operatorname{Hom}^\bullet(-, -) : \mathsf{D}^-(\mathcal{A})^{\mathrm{op}} \times \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathsf{AbGrp})$$
$$\mathsf{R}_2\mathsf{R}_1 \operatorname{Hom}^\bullet(-, -) : \mathsf{D}^-(\mathcal{A})^{\mathrm{op}} \times \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathsf{AbGrp})$$

are unique up to a unique canonical natural isomorphism.

Proof. Let ε_2 and ε_1 be the natural transformations associated to the derived functors $R_2 \operatorname{Hom}^\bullet(X^\bullet, -)$ and $\mathsf{R}_1 \operatorname{Hom}(-, Y^\bullet)$. These induces natural transformations

$$\varepsilon_2 : Q_{\mathcal{A}} \circ \operatorname{Hom}^\bullet(-, -) \to \mathsf{R}_1\mathsf{R}_2 \operatorname{Hom}^\bullet(-, -) \circ (Q_{\mathcal{A}}^{\mathrm{op}} \times Q_{\mathcal{A}})$$
$$\varepsilon_1 : Q_{\mathcal{A}} \circ \operatorname{Hom}^\bullet(-, -) \to \mathsf{R}_2\mathsf{R}_1 \operatorname{Hom}^\bullet(-, -) \circ (Q_{\mathcal{A}}^{\mathrm{op}} \times Q_{\mathcal{A}})$$

By the universal property of derived functors, there exists unique natural transformations

$$\eta_2 : \mathsf{R}_1\mathsf{R}_2 \operatorname{Hom}^\bullet(-, -) \to \mathsf{R}_2\mathsf{R}_1 \operatorname{Hom}^\bullet(-, -)$$
$$\eta_1 : \mathsf{R}_2\mathsf{R}_1 \operatorname{Hom}^\bullet(-, -) \to \mathsf{R}_1\mathsf{R}_2 \operatorname{Hom}^\bullet(-, -)$$

such that

$$(\eta_2)_{(Q_{\mathcal{A}}^{\mathrm{op}} \times Q_{\mathcal{A}})} \circ \varepsilon_2 = \varepsilon_1$$
$$(\eta_1)_{(Q_{\mathcal{A}}^{\mathrm{op}} \times Q_{\mathcal{A}})} \circ \varepsilon_1 = \varepsilon_2$$

But the universal property of derived functors then implies that

$$\eta_1 \circ \eta_2 = \mathrm{id}_{\mathsf{R}_1 \mathsf{R}_2 \mathrm{Hom}^\bullet(-,-)}$$
$$\eta_2 \circ \eta_1 = \mathrm{id}_{\mathsf{R}_2 \mathsf{R}_1 \mathrm{Hom}^\bullet(-,-)}$$

so that η_1 and η_2 are mutually inverse natural transformations. □

Remark. From now on, we shall simply write $\mathsf{R}\,\mathrm{Hom}^\bullet(-,-)$ to denote the derived functors of inner hom. In the case that $\mathcal{A}$ has enough injectives and projectives, the above Proposition ensures that this notation is not ambiguous.

Definition 6.2.4. Suppose that $\mathcal{A}$ has enough injectives (or projectives or both) and let $X^\bullet \in \mathrm{ob}\,\mathsf{D}(\mathcal{A})^{\mathrm{op}}$ and $Y^\bullet \in \mathrm{ob}\,\mathsf{D}^+(\mathcal{A})$. We define the ***n^th^*** Ext **group** of $X^\bullet$ and $Y^\bullet$ to be

$$\mathsf{Ext}^n(X^\bullet, Y^\bullet) = \mathsf{R}^n\,\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)$$

Proposition 6.2.5. *Suppose that $\mathcal{A}$ has enough injectives (or projectives or both) and let $X^\bullet \in \mathrm{ob}\,\mathsf{D}(\mathcal{A})^{\mathrm{op}}$ and $Y^\bullet \in \mathrm{ob}\,\mathsf{D}^+(\mathcal{A})$. Then*

$$\mathsf{Ext}^n(X^\bullet, Y^\bullet) = \mathsf{D}(\mathcal{A})(X^\bullet, Y[n]^\bullet)$$

Proof. Fix an injective resolution $I^\bullet$ of $Y^\bullet$. Then

$$\mathsf{D}(\mathcal{A})(X^\bullet, Y[n]^\bullet) = \mathsf{D}(\mathcal{A})(X^\bullet, I[n]^\bullet)$$

We first claim that

$$\mathsf{D}(\mathcal{A})(X^\bullet, I^\bullet) = \mathsf{K}(\mathcal{A})(X^\bullet, I^\bullet)$$

To this end, fix a morphism $\alpha \in \mathsf{D}(\mathcal{A})(X^\bullet, I^\bullet)$ represented by a roof $s^{-1}f : X^\bullet \to Z^\bullet \to I^\bullet$. Since $I^\bullet$ is bounded from below, a similar argument to the proof of Proposition 4.5.5 allows us to assume that $Z^\bullet$ is bounded below. Replacing $Z^\bullet$ with an injective resolution of $Z^\bullet$ if necessary, we may assume that $Z^\bullet$ is injective. But then Corollary 5.4.3 implies that s is an isomorphism in $\mathsf{K}(\mathcal{A})$. Hence the map sending α to $s^{-1}f$ defines the desired isomorphism.

In light of this, we have that

$$\mathsf{D}(\mathcal{A})(X^\bullet, Y[n]^\bullet) = \mathsf{D}(\mathcal{A})(X^\bullet, I[n]^\bullet) = \mathsf{K}(\mathcal{A})(X^\bullet, I[n]^\bullet)$$

On the other hand, Part 3 of Lemma 6.1.2 yields

$$\mathsf{D}(\mathcal{A})(X^\bullet, Y[n]^\bullet) = \mathsf{K}(\mathcal{A})(X^\bullet, I[n]^\bullet) = H^n(\mathrm{Hom}^\bullet(X^\bullet, I^\bullet))$$

But by the definition of the right derived functor, we have

$$\mathsf{D}(\mathcal{A})(X^\bullet, Y[n]^\bullet) = H^n(\mathrm{Hom}^\bullet(X^\bullet, I^\bullet)) = \mathsf{R}^n\,\mathrm{Hom}(X^\bullet, Y^\bullet)$$

□

7 Tensor Abelian Categories

7.1 Definitions and Properties

Definition 7.1.1. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is **monoidal** if it comes equipped with the following data

1. A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the **tensor** functor.
2. A distinguished object $I \in \operatorname{ob}\mathcal{C}$ called the **unit**.
3. A natural isomorphism α, called the **associator**, with components $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$.
4. Natural isomorphisms λ and ρ, called the **left and right unitor** respectively, with components $\lambda_X : I \otimes X \cong X$ and $\rho_X : X \cong X \otimes I$.

such that the following diagrams commute

$$
\begin{array}{ccccc}
((W \otimes X) \otimes Y) \otimes Z & \xrightarrow{\alpha_{W,X,Y} \otimes \mathrm{id}_Z} & (W \otimes (X \otimes Y)) \otimes Z & \xrightarrow{\alpha_{W,X\otimes Y,Z}} & W \otimes ((X \otimes Y) \otimes Z) \\
\downarrow{\scriptstyle \alpha_{W\otimes X,Y,Z}} & & & & \downarrow{\scriptstyle \mathrm{id}_W \otimes \alpha_{X,Y,Z}} \\
(W \otimes X) \otimes (Y \otimes Z) & & \xrightarrow{\qquad\qquad \alpha_{W,X,Y\otimes Z} \qquad\qquad} & & W \otimes (X \otimes (Y \otimes Z))
\end{array}
$$

$$
\begin{array}{ccc}
(X \otimes I) \otimes Y & \xrightarrow{\alpha_{X,I,Y}} & X \otimes (I \otimes Y) \\
& {\scriptstyle \rho_X \otimes \mathrm{id}_Y} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_X \otimes \lambda_Y} & \\
& X \otimes Y &
\end{array}
$$

Moreover, we say that $\mathcal{C}$ is **symmetric** if there exists a natural isomorphism γ, called the **braiding**, with components $\gamma_{X,Y} : X \otimes Y \to Y \otimes X$ such that $\lambda_X \circ \gamma_{X,I} = \rho_X$, $\gamma_{Y,X} \circ \gamma_{X,Y} = \mathrm{id}_{X\otimes Y}$ and such that the diagram

$$
\begin{array}{ccc}
(X \otimes Y) \otimes Z & \xrightarrow{\gamma_{X,Y} \otimes \mathrm{id}_Z} & (Y \otimes X) \otimes Z \\
\downarrow{\scriptstyle \alpha_{X,Y,Z}} & & \downarrow{\scriptstyle \alpha_{Y,X,Z}} \\
X \otimes (Y \otimes Z) & & Y \otimes (X \otimes Z) \\
\downarrow{\scriptstyle \gamma_{X,Y\otimes Z}} & & \downarrow{\scriptstyle \gamma_{Y,X\otimes Z}} \\
(Y \otimes Z) \otimes X & \xrightarrow{\alpha_{Y,Z,X}} & Y \otimes (Z \otimes X)
\end{array}
$$

commutes.

Definition 7.1.2. Let $\mathcal{A}$ be an additive category. We say that $\mathcal{A}$ is a tensor category if it is a symmetric monoidal category whose tensor functor is additive in both arguments. If, in addition, $\mathcal{A}$ is abelian then we require that its tensor functor preserves finite colimits in both arguments.

Example 7.1.3. Let R be a commutative ring with unit. Then Mod_R is a tensor abelian category. Moreover, if X is a topological space then the category of Mod_R-valued sheaves on X is a tensor abelian category.

From now on, $\mathcal{A}$ will be an abelian tensor category.

Definition 7.1.4. We say that $F \in \operatorname{ob} \mathcal{A}$ is **flat** if the functor $- \otimes F$ is exact.

Proposition 7.1.5. *Let $\mathfrak{F}$ the collection of all flat objects in $\mathcal{A}$. Then $\mathfrak{F}$ is admissible.*

Proof. It is immediate that any object isomorphic to a flat object is necessarily flat. The fact that the direct sum of flat modules is flat follows from the fact that the tensor functor is additive. □

Definition 7.1.6. Let $X^\bullet, Y^\bullet \in \operatorname{ob} \mathsf{Com}(\mathcal{A})$. Then we define the **tensor product** of $X^\bullet$ and $Y^\bullet$ to be the complex $X^\bullet \otimes Y^\bullet \in \mathsf{Com}(\mathcal{A})$ given by the data

$$(X^\bullet \otimes Y^\bullet)^i = \bigoplus_{p+q=i} X^p \otimes Y^q$$
$$(d^i_{X^\bullet \otimes Y^\bullet})_{p+q=i} = (d^p_X \otimes \mathrm{id}^q_{Y^\bullet} + (-1)^p(\mathrm{id}^p_{X^\bullet} \otimes d^q_{Y^\bullet}))_{p+q=i}$$

Given morphisms $u \in \mathsf{Com}(\mathcal{A})(X^\bullet, X'^\bullet)$ and $v \in \mathsf{Com}(\mathcal{A})(Y^\bullet, Y'^\bullet)$, we define a morphism of complexes

$$u \otimes v : X^\bullet \otimes Y^\bullet \to X'^\bullet \otimes Y'^\bullet$$

componentwise. This then defines the bifunctor **tensor product**

$$\otimes : \mathsf{Com}(\mathcal{A}) \times \mathsf{Com}(\mathcal{A}) \to \mathsf{Com}(\mathcal{A})$$

Proposition 7.1.7. *Let I be the unit of $\mathcal{A}$. Then the tensor product of complexes, together with $I[0]$ acting as the unit, endows $\mathsf{Com}(\mathcal{A})$ with the structure of an abelian tensor category.*

Proof. We omit this simple yet tedious proof. Each axiom of a symmetric monoidal category follows from the corresponding axiom for $\mathcal{A}$. □

7.2 Derived Functor of $\otimes$

Proposition 7.2.1. $\otimes : \mathsf{Com}(\mathcal{A}) \to \mathsf{Com}(\mathcal{A})$ *preserves homotopy equivalences and thus descends to an additive functor*

$$\otimes : \mathsf{K}(\mathcal{A}) \times \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$$

which endows $\mathsf{K}(\mathcal{A})$ with the stucture of an abelian tensor category.

Proof. It suffices to show that $\otimes$ preserves homotopy equivalence in the first argument, the argumentation for the second argument follows similarly. Let $u : X^\bullet \to X'^\bullet$ be a null-homotopic morphism. We need to show that $u \otimes \mathrm{id}_{Y^\bullet}$ is null-homotopic. Fix $k \in \mathcal{A}^{-1}(X^\bullet, X'^\bullet)$ such that $u = d_Y k + k d_X$. We claim that $k \otimes \mathrm{id}_{Y^\bullet}$ is a homotopy operator from $u \otimes \mathrm{id}_{Y^\bullet}$ to 0. Indeed, we have

$$\begin{aligned}
u^p \otimes \mathrm{id}^q_{Y^\bullet} &= (d^{p-1}_{X'} k^p + k^{p+1} d^p_X) \otimes \mathrm{id}^q_{Y^\bullet} \\
&= d^{p-1}_{X'} k^p \otimes \mathrm{id}^q_{Y^\bullet} + k^{p+1} d^p_X \otimes \mathrm{id}^q_{Y^\bullet} \\
&= d^{p-1}_{X'} k^p \otimes \mathrm{id}^q_{Y^\bullet} + (-1)^{p-1}(k^p \otimes d^q_Y) - (-1)^{p-1}(k^p \otimes d^q_Y) + k^{p+1} d^p_X \otimes \mathrm{id}^q_{Y^\bullet} \\
&= (d^{p-1}_{X'} \otimes \mathrm{id}^q_{Y^\bullet} + (-1)^p(\mathrm{id}^{p-1}_{X'^\bullet} \otimes d^q_Y))(k^p \otimes \mathrm{id}^q_{Y^\bullet}) \\
&\quad + (-1)^p(k^p \otimes d^q_Y) + k^{p+1} d^p_X \otimes \mathrm{id}^q_{Y^\bullet} \\
&= (d^{i-1}_{X'^\bullet \otimes Y^\bullet})_{p,q}(k^p \otimes \mathrm{id}^q_{Y^\bullet}) + (-1)^p(k^p \otimes \mathrm{id}^{q+1}_{Y^\bullet})(\mathrm{id}^p_{X^\bullet} \otimes d^q_Y) + k^{p+1} d^p_X \otimes \mathrm{id}^q_{Y^\bullet} \\
&= (d^{i-1}_{X'^\bullet \otimes Y^\bullet})_{p,q}(k^p \otimes \mathrm{id}^q_{Y^\bullet}) + (-1)^p(k^{p+1} \otimes \mathrm{id}^q_{Y^\bullet})(\mathrm{id}^p_{X^\bullet} \otimes d^q_Y) + k^{p+1} d^p_X \otimes \mathrm{id}^q_{Y^\bullet} \\
&= (d^{i-1}_{X'^\bullet \otimes Y^\bullet})_{p,q}(k^p \otimes \mathrm{id}^q_{Y^\bullet}) + (k^{p+1} \otimes \mathrm{id}^q_{Y^\bullet})(d^i_{X^\bullet \otimes Y^\bullet})_{p,q}
\end{aligned}$$

as required. □

Lemma 7.2.2. *Let $Z^\bullet \in \operatorname{ob} \mathsf{Com}(\mathcal{A})$ be a complex and $u \in \mathsf{Com}(\mathcal{A})(X^\bullet, Y^\bullet)$ a morphism. Then*

$$C(u \otimes \mathrm{id}_{Z^\bullet}) = C(u)^\bullet \otimes Z^\bullet$$

Proof. First note that we have equalities of functors $T(Z^\bullet \otimes -) = Z^\bullet \otimes T(-)$ and $T(- \otimes Z^\bullet) = T(-) \otimes Z^\bullet$. Hence

$$\begin{aligned} C(u \otimes \mathrm{id}_{Z^\bullet})^\bullet &= T(X^\bullet \otimes Z^\bullet) \oplus (Y^\bullet \otimes Z^\bullet) \\ &= (X[1]^\bullet \oplus Y^\bullet) \otimes Z^\bullet \\ &= C(u)^\bullet \otimes Z^\bullet \end{aligned}$$

so that the two complexes are componentwise isomorphic. It follows immediately that this is in fact a morphism of complexes hence the complexes are isomorphic. □

Proposition 7.2.3. *$\otimes : \mathsf{K}(\mathcal{A}) \times \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$ is a bi-δ-functor.*

Proof. It is clear that $\otimes$ commutes with shift functors. The fact that $\otimes$ sends triangles to triangles follows from Lemma 7.2.2. □

Theorem 7.2.4. *Suppose that $\mathcal{A}$ has enough left flat objects and let $\mathfrak{F}$ be the collection of flat objects in $\mathcal{A}$. Then for all $Y^\bullet \in \operatorname{ob} \mathsf{K}^-(\mathcal{A})$, $\mathfrak{F}$ is adapted to $- \otimes Y^\bullet$. Dually, $\mathfrak{F}$ is adapted to $X^\bullet \otimes -$.*

Proof. Proof omitted. See [Bor+87, Lemma I.11.5] and [Har66, Lemma I.4.1]. □

Definition 7.2.5. Suppose that $\mathcal{A}$ has enough left flat objects and let $\mathfrak{F}$ be the collection of flat objects in $\mathcal{A}$. Given an object $Y^\bullet \in \operatorname{ob} \mathsf{K}(\mathcal{A})$, the functor $- \otimes Y^\bullet$ is a δ-functor. By Theorem 7.2.4, $\mathfrak{F}$ is adapted to $- \otimes Y^\bullet$ (considered as a functor restricted to $\mathsf{K}^-(\mathcal{A})$) so it admits a left-derived functor which we denote

$$\mathsf{L}_1(- \otimes Y^\bullet) : \mathsf{D}^-(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$$

to signifiy we are deriving the functor in the first argument. This induces a bifunctor

$$\mathsf{L}_1(- \otimes -) : \mathsf{D}^-(\mathcal{A}) \times \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$$

Given $X^\bullet \in \operatorname{ob} \mathsf{D}^-(\mathcal{A})$, we then get a functor

$$\mathsf{L}_1(X^\bullet \otimes -) : \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$$

Now suppose that $s : X^\bullet \to X'^\bullet$ is a quasi-isomorphism and $F^\bullet$ a left flat resolution of $Y^\bullet$. Then $\mathsf{L}_1(s, Y^\bullet) = s \otimes F^\bullet$ and the morphism

$$s \otimes F^\bullet : X^\bullet \otimes Y^\bullet \to X'^\bullet \otimes Y^\bullet$$

is a quasi-isomorphism since $\mathfrak{F}$ is adapted to $- \otimes Y^\bullet$ and sends acyclic complexes to acyclic complexes. hence by the universal property of the localisation functor $Q_{\mathcal{A}} : \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$, we obtain a unique functor

$$\mathsf{L}_2\mathsf{L}_1(- \otimes Y^\bullet) : \mathsf{D}^-(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$$

which induces a bifunctor

$$\mathsf{L}_2\mathsf{L}_1(- \otimes -) : \mathsf{D}^-(\mathcal{A}) \times \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$$

Dually, we can construct a unique functor

$$\mathsf{L}_1\mathsf{L}_2(- \otimes -) : \mathsf{D}(\mathcal{A}) \times \mathsf{D}^-(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$$

Proposition 7.2.6. *Suppose that $\mathcal{A}$ has enough left flat objects. Then the functors*

$$\mathsf{L}_2\mathsf{L}_1(-\otimes -) : \mathsf{D}^-(\mathcal{A}) \times \mathsf{D}^-(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$$
$$\mathsf{L}_1\mathsf{L}_2(-\otimes -) : \mathsf{D}^-(\mathcal{A}) \times \mathsf{D}^-(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$$

are unique up to unique canonical isomorphism.

Proof. This follows the exact same proof as for Proposition 6.2.3. □

Remark. From now on, we shall simply write $\overset{\mathsf{L}}{\otimes}$ to denote the derived functors of $\otimes$. The above Proposition ensures that this notation is not ambiguous.

Proposition 7.2.7. *Suppose that $\mathcal{A}$ has enough left flat objects. Then $\overset{\mathsf{L}}{\otimes}$ endows $\mathsf{D}^-(\mathcal{A})$ with the structure of an abelian tensor category.*

Proof. We omit this this simple yet tedious proof. Each axiom of a symmetric monoidal category follows from the corresponding axiom for $\mathsf{K}^-(\mathcal{A})$ together with the universal property of $\overset{\mathsf{L}}{\otimes}$. □

Definition 7.2.8. Suppose that $\mathcal{A}$ has enough left flat objects and let $X^\bullet, Y^\bullet \in \operatorname{ob} \mathsf{D}^-(\mathcal{A})$. We define the ***n***

$$\mathsf{Tor}_n(X^\bullet, Y^\bullet) = H^{-n}(X^\bullet \overset{\mathsf{L}}{\otimes} Y^\bullet)$$

8 Spectral Sequences

Throughout this section, $\mathcal{A}$ will be an abelian category.

8.1 Definitions

Definition 8.1.1. Let $\mathcal{A}$ be an abelian category and $X \in \operatorname{ob} \mathcal{A}$ an object. A **descending filtration** of X is a chain of subobjects

$$\cdots \subseteq F^{p+1}(X) \subseteq F^p(X) \subseteq F^{p-1}(X) \subseteq \ldots$$

Should they exist, we write

$$\inf_p F^p(X) = \bigcap_p F^p(X)$$
$$\sup_p F^p(X) = \bigcup_p F^p(X)$$

We say that F^p is **separated** if $\inf_p F^p(X) = 0$ and **coseparated** if $\sup_p F^p(X) = X$. Moreover, we say that F^p is **discrete** if there exists $p \in \mathbb{Z}$ such that $F^p(X) = 0$ and **codiscrete** if there exists $p \in \mathbb{Z}$ such that $F^p(X) = X$.

Definition 8.1.2. We define a **bigraded collection** in $\mathcal{A}$ to be a collection of objects $\{E^{p,q}\}_{p,q\in\mathbb{Z}}$. A **bigraded differential of bidegree (s,t)** d on a bigraded collection is a collection of morphisms $d^{p,q} : E^{p,q} \to E^{p+s,q+t}$ such that the composition of any two consecutive differentials is the zero map. A **differential bigraded collection** is a choice of bigraded collection together with a bigraded differential which we shall often write as $(E^{p,q}, d^{p,q})_{p,q\in\mathbb{Z}}$ or just (E, d).

Definition 8.1.3. Let (E, d) be a differential bigraded collection. Given $s \in \mathbb{Z}$, we say that (E, d) is **cohomological of degree *s*** if d is of bidegree $(s, 1-s)$. We define the **cohomology** of a cohomological differential bigraded collection to be

$$H^{p,q}(E, d) = \ker(d^{p,q}) / \operatorname{im} d^{p-s,q+s-1}$$

Remark. We can also make the dual definition which insists that a **homological** differential bigraded collection (E, d) has bidegree $(-s, s-1)$.

Definition 8.1.4. Let $a \geq 0$ be an integer. We define a **cohomology book *E* starting at page *a*** to be a structure consisting of the following data

1. For each $r \geq a$ a cohomological differential bigraded collection (E_r, d_r) of degree r called the ***r^th* page** of E.

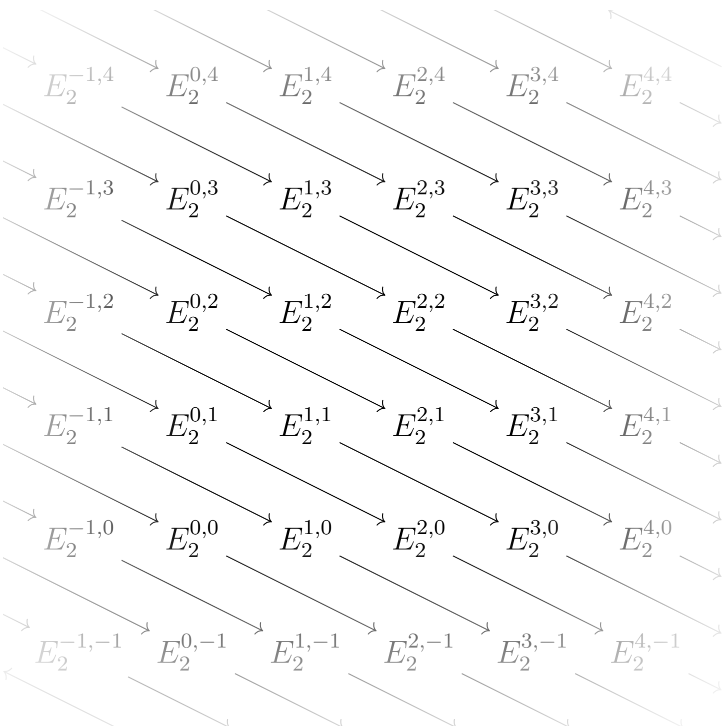
2. For each $r \geq a$ an isomorphism called the ***r^th* animation**

$$\alpha_r^{p,q} : H^{p,q}(E_r, d_r) \to E_{r+1}^{p,q}$$

For notational convenience, we will often write $Z_{r+1}(E_r^{p,q}) = \ker(d_r^{p,q})$ and $B_{r+1}(E_r^{p,q}) = \operatorname{im}(d_r^{p-r,q+r-1})$. Moreover, we will usually suppress this isomorphism and assume that the objects are simply equal.

Remark. The naming of a cohomology book suggests a useful visual aid in order to keep track of the various pieces of data. A cohomology book can be viewed as a (physical) book whose every page is a $\mathbb{Z}^2$ coordinate grid. On the $(p, q)^{th}$ coordinate of the r^{th} page is placed the object $E_r^{p,q}$. The various objects on the r^{th} page are connected via the differential. Applying the collection of animations α_r to the r^{th} page is akin to turning the page of the book - thereby 'animating' it in the style of a flip book. The following animation[4] encapsulates this visualisation

Page 2



[4] Note that this animation (probably) only works in Adobe reader and similar software

Lemma 8.1.5. *Let $\mathcal{A}$ be an abelian category. Consider the diagram*

$$\begin{array}{ccc} A' & \xrightarrow{\alpha} & A \\ & & \big\downarrow{\scriptstyle\beta} \\ B' & \xrightarrow{\gamma} & B \end{array}$$

where γ is monic. Then this diagram can be completed to a pullback if and only if α is the kernel of $\operatorname{coker}(\gamma) \circ \beta$.

Proof. First suppose that the diagram completes to a pullback

$$\begin{array}{ccc} A' & \xrightarrow{\alpha} & A \\ \big\downarrow{\scriptstyle\delta} & & \big\downarrow{\scriptstyle\beta} \\ B' & \xrightarrow{\gamma} & B \end{array}$$

We need to show that α equalises $\operatorname{coker}(\gamma) \circ \beta$ with 0 and is universal amongst such morphisms. Since the square is a pullback, we have that

$$\operatorname{coker}(\gamma) \circ \beta \circ \alpha = \operatorname{coker}(\gamma) \circ \gamma \circ \delta = 0$$

by the definition of $\operatorname{coker}(\gamma)$. We must now show that α satisfies the universal property of a kernel. Suppose that $\alpha' : Z \to A$ also equalises $\operatorname{coker}(\gamma) \circ \beta$ with 0. Then, clearly, $\beta \circ \alpha'$ equalises $\operatorname{coker}(\gamma)$ with 0. Hence $\beta \circ \alpha'$ factors through $\ker(\operatorname{coker}(\gamma))$. But this is just γ since, in any abelian category, every monomorphism is the cokernel of its kernel. We thus see that there exists some $l : Z \to B'$ such that $\gamma \circ l = \beta \circ \alpha'$. By the universal property of the pullback there thus exists a unique $k : Z \to A'$ such that $\alpha' = \alpha \circ k$. This gives the desired unique factorisation in the universal property of the kernel.

Conversely, suppose that α is the kernel of $\operatorname{coker}(\gamma) \circ \beta$. We need to show that there exists some $\delta : A' \to B'$ completing the square to a pullback. As before, we see that $\beta \circ \alpha$ equalises $\operatorname{coker}(\gamma)$ with 0 and so it necessarily factors through $\ker(\operatorname{coker}(\gamma)) = \gamma$. There thus exists some $\delta : A' \to B'$ such that $\gamma \circ \delta = \beta \circ \alpha$. We claim that the pair (δ, α) satisfies the universal property of a pullback. To this end, suppose that $\sigma : Z \to A$ and $\tau : Z \to B'$ are such that $\beta \circ \sigma = \gamma \circ \tau$. We need to exhibit a unique $k : Z \to A'$ such that $\sigma = \alpha \circ k$ and $\tau = \delta \circ k$.

Observe that

$$\beta \circ \sigma = \gamma \circ \tau \implies \operatorname{coker}(\gamma) \circ \beta \circ \sigma = \operatorname{coker}(\gamma) \circ \gamma \circ \tau = 0$$

By the universal property of $\alpha = \ker(\operatorname{coker}(\gamma) \circ \beta)$, there exists a unique $k : Z \to A'$ such that $\sigma = \alpha \circ k$. Note now that

$$\gamma \circ \delta = \beta \circ \alpha \implies \gamma \circ \delta \circ k = \beta \circ \alpha \circ k \implies \gamma \circ \delta \circ k = \beta \circ \sigma = \gamma \circ \tau$$

Since γ is monic it follows that $\delta \circ k = \tau$ and so the the pair (δ, α) is universal. □

Definition 8.1.6. Let E be a cohomology book starting at page $a \geq 0$. Given $r \geq a$ and $k \geq r + 11$, we recursively define subobjects $Z_k^{p,q}(E_r^{p,q})$ and $B_k^{p,q}(E_r^{p,q})$ of $E_r^{p,q}$ as follows. Note that we have a relation

$$E_r^{p,q} \twoheadrightarrow E_r^{p,q}/B_{r+1}(E_r^{p,q}) \hookleftarrow Z_{r+1}(E_r^{p,q})/B_{r+1}(E_r^{p,q}) \xrightarrow{\alpha_r^{p,q}} E_{r+1}^{p,q}$$

We define $Z_k^{p,q}(E_r^{p,q})$ and $B_k^{p,q}(E_r^{p,q})$ to be the subobjects of $E_r^{p,q}$ corresponding to the inverse images of $Z_k^{p,q}(E_{r+1}^{p,q})$ and $B_k^{p,q}(E_{r+1}^{p,q})$ as subobjects of $E_{r+1}^{p,q}$ respectively. Clearly, $B_k^{p,q}(E_r^{p,q}) \subseteq Z_k^{p,q}(E_r^{p,q})$ so Lemma 8.1.5 allows us to form the pullback

$$\begin{array}{ccc} B_k(E_{r+1}^{p,q}) & \longrightarrow & Z_k(E_r^{p,q}) \\ \downarrow & & \downarrow \\ B_k(E_{r+1}^{p,q}) & \longrightarrow & Z_k(E_{r+1}^{p,q}) \end{array}$$

where the horizontal maps are monic and, since epimorphisms are stable under pullback in an abelian category, the vertical maps are epimorphisms. We thus have induced isomorphisms on the cokernels

$$Z_k(E_r^{p,q})/B_k(E_r^{p,q}) \cong Z_k(E_{r+1}^{p,q})/B_k(E_{r+1}^{p,q})$$

for each $k \geq r+1$ which recursively provides us with canonical isomorphisms

$$Z_k(E_r^{p,q})/B_k(E_r^{p,q}) \cong E_k^{p,q}$$

for all $k \geq r+1$. Setting $B_r(E_r^{p,q}) = 0$ and $Z_r(E_r^{p,q}) = E_r^{p,q}$, we thus have a chain of inclusions

$$0 = B_r(E_r^{p,q}) \subseteq B_{r+1}(E_r^{p,q}) \subseteq \cdots \subseteq Z_{r+1}(E_r^{p,q}) \subseteq Z_r(E_r^{p,q}) = E_r^{p,q}$$

Definition 8.1.7. Let E be a cohomology book starting at page $a \geq 0$ in $\mathcal{A}$. We say that E is a **spectral sequence** if it comes equipped with the following data

1. Subobjects $Z_\infty(E_a^{p,q})$ and $B_\infty(E_a^{p,q})$ of $E_a^{p,q}$ such that for all $k \geq a$ we have $Z_k(E_a^{p,q}) \subseteq Z_\infty(E_r^{p,q})$ and $B_k(E_a^{p,q}) \subseteq B_\infty(E_r^{p,q})$. We set $E_\infty^{p,q} = Z_\infty(E_a^{p,q})/B_\infty(E_a^{p,q})$ and call the bigraded collection $\{ E_\infty^{p,q} \}$ the **page at infinity** of E.

2. A collection of objects $\{ E^n \}_{n\in\mathbb{Z}}$ called the **limit** of E such that each E^n comes equipped with a descending filtration $\{ F^p(E^n) \}_{p\in\mathbb{Z}}$. We let

$$\mathrm{gr}_p(E^n) = F^p(E^n)/F^{p+1}(E^n)$$

be the $\boldsymbol{p^{th}}$ **graded component** of E^n.

3. For each $p, q \in \mathbb{Z}$ an isomorphism

$$\beta^{p,q} : E_\infty^{p,q} \to \mathrm{gr}_p(E^{p+q})$$

We will often write $E_r^{p,q} \Rightarrow_r E^{p+q}$ to indicate that the spectral sequence E converges to the limit E^n.

Definition 8.1.8. Let E be a spectral sequence starting at page $a \geq 0$ in $\mathcal{A}$. We say that E is **weakly convergent** if $B_\infty(E_a^{p,q}) = \sup_k B_k(E_a^{p,q})$ and $Z_\infty = \inf_k Z_k(E_k^{p,q})$ (assuming these exist). If E is weakly convergent then we say it is **regular** if

1. For each $p, q \in \mathbb{Z}$ the descending sequence $\{ Z_k(E_a^{p,q}) \}_{k\geq a}$ **stabilises**. In other words, $Z_k(E_a^{p,q}) = Z_{k+1}(E_a^{p,q})$ for sufficiently large k. In this case we then have that $Z_\infty(E_a^{p,q}) = Z_k(E_a^{p,q})$.

2. For each $n \in \mathbb{Z}$, the filtration $\{ F^p(E^n) \}_{p\in\mathbb{Z}}$ is discrete and coseparated.

We say that E is **coregular** if it is weakly convergent and

1. For each $p, q \in \mathbb{Z}$ the ascending sequence $\{ B_k(E_a^{p,q}) \}_{k \geq a}$ **stabilises**. In this case we then have that $B_\infty(E_a^{p,q}) = B_k(E_a^{p,q})$.

2. For each $n \in \mathbb{Z}$, the filtration $\{ F^p(E^n) \}_{p \in \mathbb{Z}}$ is codiscrete.

Finally, we say that E is **biregular** if it is both regular and coregular.

Proposition 8.1.9. *Let E be a biregular spectral sequence starting at page $a \geq 0$ in $\mathcal{A}$. If for some $r \geq a$ and $p, q \in \mathbb{Z}$ we have that $E_r^{p,q} = 0$ then $E_\infty^{p,q} = 0$.*

Proof. This is immediate from the definitions. □

Definition 8.1.10. Let E be a biregular spectral sequence starting at page $a \geq 0$ in $\mathcal{A}$. We say that E **degenerates** on page $r \geq a$ if for all $p, q \in \mathbb{Z}$ we have that $d_r^{p,q} = 0$.

8.2 The Spectral Sequence of a Filtered Complex

Definition 8.2.1. Let $X^\bullet \in \operatorname{ob} \mathsf{Com}(\mathcal{A})$ be a complex. A **descending filtration** on $X^\bullet$ is a collection of descending filtrations $\{ F^p(X^n) \}_{p \in \mathbb{Z}}$ for each $n \in \mathbb{Z}$ such that $d_X(F^p(X^n)) \subseteq F^p(X^{n+1})$.

Let $X^\bullet$ be a filtered complex. Before we continue we present an informal discussion on how the filtration gives us the notion of 'approximate cocycles'. Fix $i \in \mathbb{Z}$. Given an element $x \in X^i$, let $p \in \mathbb{Z}$ be an integer such that $x \in F^p(X^i)$. Then the following philosophy applies:

The larger the magnitude of p, the closer that x is to 0

Under the guise of this philosophy, let $r \geq 0$ be an integer such that $d_X^i(x) \in F^{p+r}(X^{i+1})$. Then the larger the magnitude of r, the closer that x is to being a cocycle. We formalise this philosophy in the following definition:

Definition 8.2.2. Let $X^\bullet$ be a filtered complex. Given $r \geq 0$ and $p, q \in \mathbb{Z}$ we define a subobject of $F^p(X^{p+q})$ by

$$A_r^{p,q} = F^p(X^{p+q}) \cap (d_X^{p+q})^{-1}(F^{p+r}(X^{p+q+1}))$$

Note that when $r = 0$ we have that $A_0^{p,q} = F^p(X^{p+q})$ so that we have a chain of subobjects

$$F^p(X^{p+q}) \cap \ker(d_X^{p+q}) \subseteq \cdots \subseteq A_{r+1}^{p,q} \subseteq A_r^{p,q} \subseteq \cdots \subseteq A_1^{p,q} \subseteq A_0^{p,q} = F^p(X^{p+q})$$

Note that $d_X^{p+q-1}(A_{r-1}^{p-r+1,q+r-2}) \subseteq F^{p+1}(X^{p+q})$ and we define $\ddot{A}_r^{p,q}$ to be the image under this inclusion. We now define, for each $r \geq 0$ and $p, q \geq 0$ the **approximate (r, p, q)-cocycles** and **(r, p, q)-coboundaries** to be the quotient objects

$$Z_r^{p,q} = \frac{A_r^{p,q} + F^{p+1}(X^{p+q})}{F^{p+1}(X^{p+q})}$$

$$B_r^{p,q} = \frac{\ddot{A}_r^{p,q} + F^{p+1}(X^{p+q})}{F^{p+1}(X^{p+q})}$$

Since $Z_r^{p,q}$ and $B_r^{p,q}$ are canonically subobjects of $F^p(X^{p+q})/F^{p+1}(X^{p+q})$ and $B_r^{p+q} \subseteq Z_r^{p,q}$, we can define the **approximate (r, p, q)-cohomology object** to be the quotient object

$$E_r^{p,q} = Z_r^{p,q} / B_r^{p,q} \cong \frac{A_r^{p,q} + F^{p+1}(X^{p+q})}{\ddot{A}_r^{p,q} + F^{p+1}(X^{p+q})}$$

Remark. Suppose that $X^\bullet$ is filtered with the trivial filtration:

$$F^i(X^\bullet) = \begin{cases} X^\bullet & \text{if } i \leq 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Then we have that

$$\begin{aligned} Z_1^{0,q} &= \frac{A_1^{0,q} + F^1(X^q)}{F^1(X^q)} = F^0(X^q) \cap (d_X^q)^{-1}(F^1(X^{q+1})) = X^q \cap (d_X^q)^{-1}(0) = \ker(d_X^q) \\ B_1^{0,q} &= \frac{\ddot{A}_1^{0,q} + F^1(X^q)}{F^1(X^q)} = d_X^{q-1}(F^0(X^{q-1}) \cap (d_X^{q-1})^{-1}(X^q) = d_X^{q-1}(X^{q-1} \cap (d_X^{q-1})^{-1}(X^q)) \\ &= \operatorname{im}(d_X^{q-1}) \end{aligned}$$

so that we recover the usual cohomology groups of $X^\bullet$.

Lemma 8.2.3. *Let $X \in \operatorname{ob} \mathcal{A}$ be an object and Y, F two subobject of X. Then the canonical morphism*

$$F \to F + Y \to (F + Y)/Y$$

is an epimorphism.

Proof. By definition, $F + Y$ is the image of the canonical map $F \oplus Y \to X$. We can, moreover, realise the projection $\pi_F : F \oplus Y \to F$ as $\operatorname{coker}(Y \to F \oplus Y)$. It is clear that the composition $F \oplus Y \to F + Y \to (F + Y)/Y$ also coequalises the map $Y \to F \oplus Y$ with 0 so, by the universal property of the cokernel π_F, there exists a unique morphism making the diagram

$$\begin{array}{ccc} F \oplus Y & \longrightarrow & F + Y \\ \downarrow & & \downarrow \\ F & \dashrightarrow & (F+Y)/Y \end{array}$$

commute. Since all the non-dotted morphisms are epimorphisms, the dotted morphism is necessarily also an epimorphism. Note that the morphism of the Lemma also makes this diagram commute so it must coincide with the dotted morphism. □

Lemma 8.2.4. *Let $X^\bullet$ be a filtered complex. For each $r \geq 0$ and $p, q \in \mathbb{Z}$, the differential d_X induces a morphism $d_r^{p,q} : A_r^{p,q} \to A_r^{p+r,q-r+1}$ such that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ and the diagram*

$$\begin{array}{ccc} A_r^{p,q} & \xrightarrow{d_r^{p,q}} & A_r^{p+r,q-r+1} \\ \downarrow & & \downarrow \\ X^{p,q} & \xrightarrow{d_X^{p+q}} & X^{p+q+1} \end{array}$$

commutes. Moreover, this morphism induces a unique morphism $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r^{p+r,q-r+1} \circ d_r^{p,q}$ and the diagram

$$\begin{array}{ccc} A_r^{p,q} & \xrightarrow{d_r^{p,q}} & A_r^{p+r,q-r+1} \\ \downarrow & & \downarrow \\ E_r^{p,q} & \xrightarrow{d_r^{p,q}} & E_r^{p+q,q-r+1} \end{array}$$

commutes where $A_r^{p,q} \to E_r^{p,q}$ is the canonical epimorphism arising from Lemma 8.2.3.

Proof. We define $d_r^{p,q} : A_r^{p,q} \to A_r^{p+r,q-r+1}$ to be the restriction of the morphism d_X^{p+q} to the subobject $A_r^{p,q}$. Once we check that the image of $d_r^{p,q}$ is contained in $A_r^{p+r,q-r+1}$ then all other assertions concerning this morphism are evident. We have that

$$d_X^{p,q}(A_r^{p,q}) \subseteq d_X^{p,q}(F^p(X^{p+q})) \cap F^{p+r}(X^{p+q+1})$$

and $A_r^{p+r,q-r+1} = F^{p+r}(X^{p+q+1}) \cap (d_X^{p+q+1})^{-1}(F^{p+2r}(X^{p+q+2}))$. Since $d_X^{p+q+1} \circ d_X^{p,q} = 0$, it is clear that $d_X^{p,q}(F^p(X^{p+q})) \subseteq (d_X^{p+q+1})^{-1}(F^{p+2r}(X^{p+q+2}))$ so we get the desired inclusion.

We now define $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ to be the canonical morphism induced on the factor groups by $d_X^{p,q}$. This is not yet well-defined - we must first check that the $d_r^{p,q}$ is independent of the choice of approximate coycle representing an approximate cohomology class in $E_r^{p,q}$. It suffices to show that

$$d_r^{p,q}(\ddot{A}_r^{p,q} + F^{p+1}(X^{p+q})) \subseteq \ddot{A}_r^{p+r,q-r+1} + F^{p+r+1}(X^{p+q+1})$$

By the definition of $\ddot{A}_r^{p,q}$, it suffices to show the inclusion for $d_X^{p+q}(F^{p+1}(X^{p+q}))$. Note that

$$\ddot{A}_r^{p+r,q-r+1} = d_X^{p+q-1}(A_{r-1}^{p+1,q-1}) = d_X^{p+q}(F^{p+1}(X^{p+q}) \cap (d_X^{p+q})^{-1}(F^{p+r+1}(X^{p+q+1}))$$

and so this inclusion is clear. The fact that the second diagram commutes is now obvious by the construction of $d_r^{p,q}$. The fact that $d_r^{p,q}$ is a differential then follows immediately from the commutativity of this diagram. □

Lemma 8.2.5. *Let $X^\bullet$ be a filtered complex. Then for all $r \geq 0$ and $p, q \in \mathbb{Z}$ we have*

$$Z_{r+1}^{p,q} := \ker(d_r^{p,q}) \cong \frac{A_{r+1}^{p,q} + F^{p+1}(X^{p+q})}{\ddot{A}_r^{p,q} + F^{p+1}(X^{p+q})}$$

$$B_{r+1}^{p,q} := \operatorname{im}(d_r^{p-r,q+r-1}) \cong \frac{\ddot{A}_{r+1}^{p,q} + F^{p+1}(X^{p+q})}{\ddot{A}_r^{p,q} + F^{p+1}(X^{p+q})}$$

thereby inducing canonical isomorphisms

$$\alpha_r^{p,q} : Z_{r+1}^{p,q} / B_{r+1}^{p,q} \to E_{r+1}^{p,q}$$

Proof. Observe that, by an isomorphism theorem, we have an isomorphism

$$E_r^{p,q} = \frac{A_r^{p,q} + F^{p+1}(X^{p+q})}{\ddot{A}_r^{p,q} + F^{p+1}(X^{p+q})} \cong \frac{A_r^{p,q}}{A_r^{p,q} \cap (\ddot{A}_r^{p,q} + F^{p+1}(X^{p+q}))} = \frac{A_r^{p,q}}{\ddot{A}_r^{p,q} + A_{r-1}^{p+1,q-1}}$$

And by a similar argument, we have

$$\frac{A_{r+1}^{p,q} + F^{p+1}(X^{p+q})}{\ddot{A}_r^{p,q} + F^{p+1}(X^{p+q})} \cong \frac{A_{r+1}^{p,q}}{\ddot{A}_r^{p,q} + A_r^{p+1,q-1}} = \frac{A_{r+1}^{p,q}}{\ddot{A}_r^{p,q} + A_{r+1}^{p,q} \cap A_{r-1}^{p+1,q-1}} \cong \frac{A_{r+1}^{p,q} + A_{r-1}^{p+1,q-1}}{\ddot{A}_r^{p,q} + A_{r-1}^{p+1,q-1}}$$

so it suffices to show that the last quotient object in the above chain of isomorphisms is $\ker(d_r^{p,q})$. To this end, fix $a \in A_r^{p,q}$ such that $d_r^{p,q}(a) = 0 \in E_r^{p+r,q+r-1}$. That is to say,

$$d_r^{p,q}(a) \in \ddot{A}_r^{p+r,q+r-1} + A_{r-1}^{p+r+1,q-r}$$

Then there exists $b \in A_{r-1}^{p+1,q-1}$ and $c \in A_{r-1}^{p+r+1,q-r}$ such that $d_X^{p+q}(a) = d_X^{p+q}(b) + c$. We have the trivial equality

$$a = (a-b) + b$$

Clearly, $a - b \in F^p(X^{p+q})$ and $d_X^{p+q}(a-b) = c \in A_{r-1}^{p+r+1,q-r} \subseteq F^{p+r+1}(X^{p+q+1})$. This is precisely what it means for $a - b \in A_{r+1}^{p,q}$.

$$\ker(d_r^{p,q}) \subseteq \frac{A_{r+1}^{p,q} + A_{r-1}^{p+1,q-1}}{\ddot{A}_r^{p,q} + A_{r-1}^{p+1,q-1}}$$

Conversely, suppose that $a \in A_{r+1}^{p,q}$. Then, by definition, $d_X^{p+q}(a) \in F^{p+r+1}(X^{p+q+1})$ and $(d_X^{p+q+1} \circ d_X^{p+q})(a) = 0$ so that $d_X^{p+q}(a) \in A_{r-1}^{p+r+,q-r}$. Similarly, if $a \in A_{r-1}^{p+1,q-1}$ then $d_X^{p+q}(a) \in \ddot{A}^{p+r,q+r-1}$ by definition hence

$$\frac{A_{r+1}^{p,q} + A_{r-1}^{p+1,q-1}}{\ddot{A}_r^{p,q} + A_{r-1}^{p+1,q-1}} \subseteq \ker(d_r^{p,q})$$

To exhibit the image isomorphism observe that, by a similar argument as for the kernel, we have

$$\frac{\ddot{A}_{r+1}^{p,q} + F^{p+1}(X^{p+q})}{\ddot{A}_r^{p,q} + F^{p+1}(X^{p+q})} \cong \frac{\ddot{A}_{r+1}^{p,q} + A_{r-1}^{p+1,q-1}}{\ddot{A}_r^{p,q} + A_{r-1}^{p+1,q-1}}$$

Now,

$$d_r^{p-r,q+r-1}(E_r^{p-r,q+r-1}) \cong \frac{d_r^{p-r,q+r-1}(A_r^{p-r,q+r-1}) + \ddot{A}_r^{p,q} + A_{r-1}^{p+1,q-1}}{\ddot{A}_r^{p,q} + A_{r-1}^{p+1,q-1}} = \frac{\ddot{A}_{r+1}^{p,q} + \ddot{A}_r^{p,q} + A_{r-1}^{p+1,q-1}}{\ddot{A}_r^{p,q} + A_{r-1}^{p+1,q-1}}$$

Since $\ddot{A}_r^{p,q} \subseteq \ddot{A}_{r+1}^{p,q}$, we obtain the desired isomorphism. We then immediately obtain the canonical isomorphisms $\alpha_r^{p,q}$. □

Proposition 8.2.6. *Let $X^\bullet$ be a filtered complex. Then the construction (E_r, d_r) is a cohomology book starting at page 0.*

Proof. This is the content of Lemma 8.2.4 and Lemma 8.2.5. □

Theorem 8.2.7. *Let $X^\bullet$ be a filtered complex and E its associated cohomology book starting on page a. Given $r \geq 0$ and $k \geq r+1$ we have*

$$Z_k(E_a^{p,q}) = \frac{A_k^{p,q} + F^{p+1}(X^{p+q})}{\ddot{A}_r^{p,q} + F^{p+1}(X^{p+q})}$$
$$B_k(E_a^{p,q}) = \frac{\ddot{A}_k^{p,q} + F^{p+1}(X^{p+q})}{\ddot{A}_r^{p,q} + F^{p+1}(X^{p+q})}$$

Moreover, E can be endowed with the stucture of a spectral sequence by defining its page at infinity to be

$$Z_\infty(E_0^{p,q}) = \frac{\ker(d_X^{p+q}) \cap F^p(X^{p+q}) + F^{p+1}(X^{p+q})}{F^{p+1}(X^{p+q})}$$
$$B_\infty(E_0^{p,q}) = \frac{\operatorname{im}(d_X^{p+q-1}) \cap F^p(X^{p+q}) + F^{p+1}(X^{p+q})}{F^{p+1}(X^{p+q})}$$

and its limit to be $E^n := H^n(X^\bullet)$.

Proof. The isomorphisms of the k^{th} approximate cocycles and coboundaries are clear from the definitions and it follows from this that the page at infinity of E is well-defined.

There is a natural filtration on E^n defined as follows. For all $p \in \mathbb{Z}$, the canonical inclusion $F^p(X^\bullet) \to X^\bullet$ induces a morphism $H^n(F^p(X^\bullet)) \to H^n(X^\bullet)$ and we denote the image of this morphism by $F^p(E^n)$. This clearly defines a descending filtration on E^n. It is clear that

$$F^p(E^{p+q}) = \frac{\ker(d_X^{p+q}) \cap F^p(X^{p+q}) + \operatorname{im}(d_X^{p+q-1})}{\operatorname{im}(d_X^{p+q-1})}$$

and so

$$\begin{aligned}\operatorname{gr}_p(E^{p+q}) = \frac{F^p(E^{p+q})}{F^{p+1}(E^{p+q})} &\cong \frac{\ker(d_X^{p+q}) \cap F^p(X^{p+q}) + \operatorname{im}(d_X^{p+q-1})}{\ker(d_X^{p+q}) \cap F^{p+1}(X^{p+q}) + \operatorname{im}(d_X^{p+q-1})} \\ &\cong \frac{\ker(d_X^{p+q}) \cap F^p(X^{p+q})}{\ker(d_X^{p+q}) \cap F^p(X^{p+q}) \cap (\ker(d_X^{p+q}) \cap F^{p+1}(X^{p+q}) + \operatorname{im}(d_X^{p+q-1}))}\end{aligned}$$

On the other hand, we have

$$\begin{aligned}E_\infty^{p,q} = \frac{Z_\infty(E_a^{p,q})}{B_\infty(E_a^{p,q})} &\cong \frac{\ker(d_X^{p+q}) \cap F^p(X^{p+q}) + F^{p+1}(X^{p+q})}{\operatorname{im}(d_X^{p+q-1}) \cap F^p(X^{p+q}) + F^{p+1}(X^{p+q})} \\ &\cong \frac{\ker(d_X^{p+q}) \cap F^p(X^{p+q})}{\ker(d_X^{p+q}) \cap F^p(X^{p+q}) \cap (\operatorname{im}(d_X^{p+q-1}) \cap F^p(X^{p+q}) + F^{p+1}(X^{p+q}))}\end{aligned}$$

It is clear that these denominators coincide so that we have canonical isomorphisms $\beta^{p,q}$: $E_\infty^{p,q} \to \operatorname{gr}_p(E^{p+q})$ whence E is a spectral sequence. □

Proposition 8.2.8. *Let $X^\bullet$ be a filtered complex and E its associated spectral sequence starting on page a. Then for all $p, q \in \mathbb{Z}$ there exists a canonical isomorphism $E_0^{p,q} \cong F^p(X^{p+q})/F^{p+1}(X^{p+q})$ such that the diagram*

$$\begin{array}{ccc} E_0^{p,q} & \xrightarrow{\sim} & \frac{F^p(X^{p+q})}{F^{p+1}(X^{p+q})} \\ \downarrow d_0^{p,q} & & \downarrow \\ E_0^{p,q+1} & \xrightarrow{\sim} & \frac{F^p(X^{p+q})}{F^{p+1}(X^{p+q+1})} \end{array}$$

commutes.

Proof. By definition we have

$$E_0^{p,q} = \frac{A_0^{p,q} + F^{p+1}(X^{p+q})}{\ddot{A}_0^{p,q} + F^{p+1}(X^{p+q})}$$

But $A_0^{p,q} = F^p(X^{p+q})$ and $\ddot{A}_0^{p,q} \subseteq F^{p+1}(X^{p+q})$ and so the isomorphism follows immediately. It is then evident from the definition of $d_0^{p,q}$ that the diagram given in the Proposition commutes. □

8.3 First Quadrant Filtrations

Definition 8.3.1. Let $X^\bullet$ be a filtered complex such that $X^i = 0$ for all $i < 0$. We say that the filtration F^p on $X^\bullet$ is **first quadrant** if

$$F^p(X^n) = \begin{cases} 0 & \text{if } p > n \\ X^n & \text{if } p \leq 0 \end{cases}$$

If E is the spectral sequence associated to $X^\bullet$ then we also say that E is a **first quadrant spectral sequence**.

Proposition 8.3.2. *Let $X^\bullet$ be a first quadrant filtered complex. Then the spectral sequence E associated to $X^\bullet$ is biregular.*

Proof. Fix $p, q \in \mathbb{Z}$ such that $p + q \geq 0$. Then for all pages $r > q + 1$ we have that $F^{p+r}(X^{p+q+1}) = 0$ so that $A_r^{p,q} = F^p(X^{p+q}) \cap \ker(d_X^{p+q})$. Similarly, for $r \geq p + 1$ we have that $A_{r-1}^{p-r+1,q+r-2} = (d_X^{p+q-1})^{-1}(F^p(X^{p+q}))$ so that $\ddot{A}_r^{p,q} = F^p(X^{p+q}) \cap \operatorname{im}(d_X^{p+q-1})$. It is then clear from the definition of E and the filtration that E is biregular. □

Remark. The same reasoning a above applies to any bounded below complex $X^\bullet \in \operatorname{ob} \mathsf{Com}^+(\mathcal{A})$ with the appropriate filtration. It merely suffices to apply the shift functor enough times to obtain a first quadrant filtration.

Theorem 8.3.3. *Let $E_r^{p,q} \Rightarrow_r H^{p+q}(X^\bullet)$ be a first quadrant spectral sequence. Then there exists an exact sequence of low-degree terms*

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(X^\bullet) \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \longrightarrow H^2(X^\bullet)$$

called the ***five-term exact sequence*** *associated to E.*

Proof. The differential $d_2^{-1,1} : E_2^{-1,1} \to E_2^{1,0}$ is the zero map since $E_2^{-1,1} = 0$. Similarly, the differential $d_2^{1,0} : E_2^{1,0} \to E_2^{3,-1}$ is zero. The same arguments show that the differentials with domains and codomains $E_r^{1,0}$ vanish for all $r \geq 2$. Hence the spectral sequence is degenerate at $E_2^{1,0}$ and we have

$$E_2^{1,0} \cong \operatorname{gr}_1(H^1(X^\bullet)) = \frac{F^1(H^1(X^\bullet))}{F^2(H^1(X^\bullet))} = F^1(H^1(X^\bullet)) \hookrightarrow H^1(X^\bullet)$$

Now, $d_2^{-2,2} : E_2^{-2,2} \to E_2^{0,1}$ is zero and hence $E_3^{0,1}$ is the kernel of the differential $d_2^{0,1} : E_2^{0,1} \to E_2^{2,0}$. Note that since the differentials with domain and codomain $E_r^{0,1}$ are trivial for $r \geq 3$, the spectral sequence is degenerate at $E_3^{0,1}$ so we have an isomorphism

$$E_3^{0,1} \cong \operatorname{gr}_0(H^1(X^\bullet)) = \frac{H^1(X^\bullet)}{F^1(H^1(X^\bullet))}$$

so that we have a short exact sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(X^\bullet) \longrightarrow E_3^{0,1} \longrightarrow 0$$

But since $E_3^{0,1}$ is the kernel of $d_2^{0,1}$, we can replace the end of this short exact sequence with $d_2^{0,1}$ to obtain an exact sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(X^\bullet) \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0}$$

To obtain the final term of the desired exact sequence, observe that the differential $d_2^{2,0}$ is trivial and so $E_3^{2,0}$ is the cokernel of $d_2^{0,1}$. Moreover, the differentials with domains and codomains $E_r^{2,0}$ are all trivial for $r \geq 3$ so that the spectral sequence is degenerate at $E_3^{2,0}$ and we have

$$E_3^{2,0} \cong \mathrm{gr}_2(H^2(X^\bullet)) \cong \frac{F^2(H^2(X^\bullet))}{F^3(H^2(X^\bullet))} = F^2(H^2(X^\bullet)) \hookrightarrow H^2(X^\bullet)$$

Since $E_3^{2,0}$ is the cokernel of $d_2^{0,1}$ it then follows that the kernel of the composite $E_2^{2,0} \to E_3^{2,0} \hookrightarrow E_3^{2,0}$ is the image of $d_2^{0,1}$. This yields the exact sequence of the Theorem. □

Corollary 8.3.4. *Let $E_r^{p,q} \Rightarrow_r H^{p+q}(X^\bullet)$ be a first quadrant spectral sequence. Then the previous exact sequence admits an extension*

$$\begin{array}{l} 0 \longrightarrow E_2^{1,0} \longrightarrow H^1(X^\bullet) \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} \\ E_2^{2,0} \longrightarrow \ker(H^2(X^\bullet) \to E_2^{0,2}) \longrightarrow E_2^{1,1} \xrightarrow{d_2^{1,1}} \\ E_2^{3,0} \end{array}$$

called the ***seven-term exact sequence*** *associated to E.*

Proof. Note that the differentials whose domains and codomains are $E_r^{1,1}$ are trivial for all $r \geq 3$ so that we have isomorphisms

$$E_3^{1,1} \cong \mathrm{gr}_1(H^2(X^\bullet)) = \frac{F^1(H^2(X^\bullet))}{F^2(H^2(X^\bullet))} \cong \frac{F^1(H^2(X^\bullet))}{E_3^{2,0}}$$

Note that

$$E_3^{1,1} \cong \frac{\ker(d_2^{1,1} : E_2^{1,1} \to E_2^{3,0})}{\mathrm{im}(d_2 : E_2^{-1,2} \to E_2^{1,1})} = \ker(d_2^{1,1})$$

Moreover, $E_3^{2,0}$ is the cokernel of $d_2^{0,1}$ so that we get a long exact sequence

$$\begin{array}{l} 0 \longrightarrow E_2^{1,0} \longrightarrow H^1(X^\bullet) \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} \\ E_2^{2,0} \longrightarrow F^1(H^2(X^\bullet)) \longrightarrow E_2^{1,1} \xrightarrow{d_2^{1,1}} \\ E_2^{3,0} \end{array}$$

To complete the proof, therefore, it suffices to show that $F^1(H^2(X^\bullet)) \cong \ker(H^2(X^\bullet) \to E_2^{0,2})$. Note that

$$E_\infty^{0,2} \cong \mathrm{gr}_0(H^2(X^\bullet)) = \frac{H^2(X^\bullet)}{F^1(H^2(X^\bullet))}$$

so that $F^1(H^2(X^\bullet)) \cong \ker(H^2(X^\bullet) \to E_\infty^{0,2})$. The claim then follows from the fact that $E_\infty^{0,2}$ is a subobject of $E_2^{0,2}$. In particular, the differentials whose domains and codomains are $E_r^{0,2}$ are trivial for $r \geq 4$ so that the spectral sequence is degenerate at $E_4^{0,2}$. Note that $E_4^{0,2} \cong \ker(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0})$ whence $E_4^{0,2} \subseteq E_3^{0,2}$. Similarly, $E_3^{0,2} \subseteq E_2^{0,2}$. □

Corollary 8.3.5. *Let $E_r^{p,q} \Rightarrow_r H^{p+q}(X^\bullet)$ be a first quadrant spectral sequence. Suppose that the objects $E_2^{p,q}$ vanish for all $0 < q < n$. Then $E_2^{p,0} \cong H^p(X^\bullet)$ for all $p < n$ and there exists an exact sequence*

$$0 \longrightarrow E_2^{n,0} \longrightarrow H^n(X^\bullet) \longrightarrow E_2^{n-1,1} \xrightarrow{d_2^{n-1,1}} E_2^{n+1,0} \longrightarrow H^{n+1}(X^\bullet)$$

called the ***higher five-term exact sequence*** *associated to E.*

Proof. This follows from the Theorem by translating the spectral sequence. □

8.4 The Spectral Sequence of a Double Complex

Throughout this section we shall assume that $\mathcal{A}$ is a cocomplete abelian category.

Definition 8.4.1. We define a **double complex** in $\mathcal{A}$ to be a triple $(X^{\bullet,\bullet}, d_X, \partial_X)$ where $X^{\bullet,\bullet}$ is a bigraded collection,

$$d_X^{i,\bullet} : X^{i,\bullet} \to X^{i+1,\bullet}$$
$$\partial_X^{\bullet,j} : X^{\bullet,j} \to X^{\bullet,j+1}$$

are differentials and such that

$$\partial_X^{i,j+1} \circ d_X^{i,j} + d_X^{i+1,j} \circ \partial_X^{i,j} = 0$$

We will often view a double complex as a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & X^{i-1,j+1} & \xrightarrow{d_X^{i-1,j+1}} & X^{i,j+1} & \xrightarrow{d_X^{i,j+1}} & X^{i+1,j+1} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \partial_X^{i-1,j}} & & \uparrow{\scriptstyle \partial_X^{i,j}} & & \uparrow{\scriptstyle \partial_X^{i+1,j}} & & \\
\cdots & \longrightarrow & X^{i-1,j} & \xrightarrow{d_X^{i-1,j}} & X^{i,j} & \xrightarrow{d_X^{i,j}} & X^{i+1,j} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \partial_X^{i-1,j-1}} & & \uparrow{\scriptstyle \partial_X^{i,j-1}} & & \uparrow{\scriptstyle \partial_X^{i+1,j-1}} & & \\
\cdots & \longrightarrow & X^{i-1,j-1} & \xrightarrow{d_X^{i-1,j-1}} & X^{i,j-1} & \xrightarrow{d_X^{i,j-1}} & X^{i+1,j-1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

We define a **morphism** of double complexes $f : X^{\bullet,\bullet} \to Y^{\bullet,\bullet}$ to be a collection of morphisms $f^{i,j} : X^{i,j} \to Y^{i,j}$ such that $f^{i,\bullet}$ and $f^{\bullet,j}$ are morphisms of complexes for all $i, j \in \mathbb{Z}$. We denote by $\mathsf{2Com}(\mathcal{A})$ the category whose objects are the double complexes in $\mathcal{A}$ and whose morphisms are the morphisms of double complexes.

Definition 8.4.2. Let $f : X^{\bullet,\bullet} \to Y^{\bullet,\bullet}$ be a morphism of double complexes. We say that f is **null-homotopic** if there exists collections of morphisms

$$k^{ij} : X^{i,j} \to Y^{i-1,j}, \quad \kappa^{i,j} : X^{i,j} \to Y^{i,j-1}$$

such that

$$\begin{aligned} f &= d_X k + k d_X + \partial_X \kappa + \kappa \partial_X \\ 0 &= k \partial_X + \partial_X k \\ 0 &= \kappa d_X + d_X \kappa \end{aligned}$$

Given another morphism $f : X^{\bullet,\bullet} \to Y^{\bullet,\bullet}$, we say that f is **homotopic** to g if $f - g$ is null-homotopic. As before, this relation is an equivalence relation and preserves composition of morphisms so we define the **homotopy** category of $2\mathsf{Com}(\mathcal{A})$, denoted $2\mathsf{K}$, to be the one whose objects are $\operatorname{ob} 2\mathsf{Com}(\mathcal{A})$ and the morphisms are $\operatorname{mor}(\mathcal{A})$ modulo homotopy.

Definition 8.4.3. Let $X^{\bullet,\bullet}$ be a double complex, we define the **horizontal** and **vertical** cohomology objects of $X^{\bullet,\bullet}$ to be

$$\begin{aligned} H^i_{>}(X^{\bullet,j}) &= \frac{\ker(d_X^{i,j})}{\ker(d_X^{i-1,j})} \\ H^j_{\wedge}(X^{i,\bullet}) &= \frac{\ker(\partial_X^{i,j})}{\operatorname{im}(\partial_X^{i,j-1})} \end{aligned}$$

respectively. We naturally have differentials

$$\begin{aligned} H^i_{>}(\partial_X^{\bullet,j}) &: H^i_{>}(X^{\bullet,j}) \to H^i_{>}(X^{\bullet,j+1}) \\ H^j_{\wedge}(d_X^{i,\bullet}) &: H^j_{\wedge}(X^{i,\bullet}) \to H^j_{\wedge}(X^{i+1,\bullet}) \end{aligned}$$

making the cohomology objects into complexes. We denote the cohomology of these complexes by

$$H^j_{\wedge}(H^i_{>}(X^{\bullet,\bullet})) \qquad \text{and} \qquad H^i_{>}(H^j_{\wedge}(X^{\bullet,\bullet}))$$

respectively.

Definition 8.4.4. Let $X^{\bullet,\bullet}$ be a double complex. We define the **total complex** of $X^{\bullet}$ to be the one given by the data

$$\begin{aligned} \operatorname{Tot}(X)^n &= \bigoplus_{i+j=n} X^{i,j} \\ d^n_{\operatorname{Tot}(X)} \circ \nu_{i,j} &= \nu_{i+1,j} d^{i,j} + \nu_{i,j+1} \partial^{i,j} \end{aligned}$$

for $i, j \in \mathbb{Z}$ such that $i + j = n$ where $\nu_{i,j} : X^{i,j} \to \bigoplus_{i+j=n} X^{i,j}$ is the canonical inclusion.

Example 8.4.5. The mapping cone of a morphism, inner hom, and the tensor product of complexes are all examples of a total complex of a double complex.

Definition 8.4.6. Let $X^{\bullet,\bullet}$ be a double complex. We define the **canonical filtrations** of $\operatorname{Tot}(X)^{\bullet}$ to be

$$\begin{aligned} F^p(\operatorname{Tot}(X)^n) &= \bigoplus_{r \geq p} X^{r,n-r} \\ \mathcal{F}^p(\operatorname{Tot}(X)^n) &= \bigoplus_{r \geq p} X^{n-r,r} \end{aligned}$$

We denote by E and $\mathcal{E}$, respectively, the associated spectral sequences starting at page 0 of the filtrations F and $\mathcal{F}$. We shall use d_r for the differentials on the r^{th} pages of E and $\mathcal{E}$ - it will be clear from context as to which spectral sequence we are referring to.

Proposition 8.4.7. *Let $X^{\bullet,\bullet}$ be a double complex. Then for all $p, q \in \mathbb{Z}$, there are canonical isomorphisms $E_0^{p,q} \cong X^{p,q}$ and $\mathcal{E}_0^{p,q} \cong X^{q,p}$ such that the diagrams*

$$\begin{array}{ccc} E_0^{p,q} & \longrightarrow & X^{p,q} \\ \Big\downarrow d_0^{p,q} & & \Big\downarrow \partial_X^{p,q} \\ E_0^{p,q+1} & \longrightarrow & X^{p,q+1} \end{array} \qquad \begin{array}{ccc} \mathcal{E}_0^{p,q} & \longrightarrow & X^{q,p} \\ \Big\downarrow d_0^{p,q} & & \Big\downarrow d_X^{q,p} \\ \mathcal{E}_0^{p,q+1} & \longrightarrow & X^{q+1,p} \end{array}$$

commute.

Proof. By Proposition 8.2.8 we have an isomorphism

$$E_0^{p,q} \cong \frac{F^p(\mathrm{Tot}(X)^{p+q})}{F^{p+1}(\mathrm{Tot}(X)^{p+q})} = \frac{\bigoplus_{r \geq p} X^{r,p+q-r}}{\bigoplus_{r \geq p+1} X^{r,p+q-r}} \cong X^{p,q}$$

and, similarly, an isomorphism $\mathcal{E}_0^{p,q} \cong X^{q,p}$. Furthermore, it is clear from Proposition 8.2.8 that these isomorphisms make the above squares commute. □

Proposition 8.4.8. *Let $X^{\bullet,\bullet}$ be a double complex. Then for all $p, q \in \mathbb{Z}$, there are canonical isomorphisms $E_1^{p,q} \cong H_\wedge^q(X^{p,\bullet})$ and $\mathcal{E}_1^{q,p} \cong H_>^q(X^{\bullet,p})$ such that the diagrams*

$$\begin{array}{ccc} E_1^{p,q} & \xrightarrow{d_1^{p,q}} & E_1^{p+1,q} \\ \Big\downarrow \wr & & \Big\downarrow \wr \\ H_\wedge^q(X^{p,\bullet}) & \xrightarrow{H_\wedge^q(d_X^{p,\bullet})} & H_\wedge^q(X^{p+1,\bullet}) \end{array} \qquad \begin{array}{ccc} \mathcal{E}_1^{p,q} & \xrightarrow{d_1^{p,q}} & \mathcal{E}_1^{p+1,q} \\ \Big\downarrow \wr & & \Big\downarrow \wr \\ H_>^q(X^{\bullet,p}) & \xrightarrow{H_>^q(\partial_X^{\bullet,p})} & \mathcal{H}_>^q(X^{\bullet,p+1}) \end{array}$$

commute.

Proof. By Proposition 8.4.7 we have a commutative diagram

$$\begin{array}{ccccc} E_0^{p,q-1} & \xrightarrow{d_0^{p,q-1}} & E_0^{p,q} & \xrightarrow{d_0^{p,q}} & E_0^{p,q+1} \\ \Big\downarrow \wr & & \Big\downarrow \wr & & \Big\downarrow \wr \\ X^{p,q-1} & \xrightarrow{\partial_X^{p,q-1}} & X^{p,q} & \xrightarrow{\partial_X^{p,q}} & X^{p,q+1} \end{array}$$

The isomorphisms then follow immediately from the definition of a spectral sequence. It is now clear that the diagram commutes since the differential $d_{\mathrm{Tot}(X)}$ coincides with $H_\wedge^q(d_X^{p,\bullet})$ on $H_\wedge^q(X^{p,\bullet})$. A similar argument applies to the second square. □

Corollary 8.4.9. *Let $X^{\bullet,\bullet}$ be a double complex. Then we have isomorphisms*

$$E_2^{p,q} \cong H_>^p(H_\wedge^q(X^{\bullet,\bullet}))$$
$$\mathcal{E}_2^{p,q} \cong H_\wedge^p(H_>^q(X^{\bullet,\bullet}))$$

Proof. This is immediate from the definition of a spectral sequence. □

Theorem 8.4.10. *Let $X^{\bullet,\bullet}$ be a double complex. Then there exist canonical spectral sequences E and $\mathcal{E}$ associated to $X^{\bullet,\bullet}$ such that*

$$E_r^{p,q} \Rightarrow_r H^{p+q}(\mathrm{Tot}(X))$$
$$\mathcal{E}_r^{p,q} \Rightarrow_r H^{p+q}(\mathrm{Tot}(X))$$

with low degree pages

$$E_0^{p,q} \cong X^{p,q}, \quad E_1^{p,q} \cong H_\wedge^q(X^{p,\bullet}), \quad E_2^{p,q} \cong H_>^p(H_\wedge^q(X^{\bullet,\bullet}))$$
$$\mathcal{E}_0^{p,q} \cong X^{q,p}, \quad \mathcal{E}_1^{p,q} \cong H_>^q(X^{\bullet,p}), \quad \mathcal{E}_2^{p,q} \cong H_\wedge^p(H_>^q(X^{\bullet,\bullet}))$$

Proof. This is the content of Proposition 8.4.7, Proposition 8.4.8 and Corollary 8.4.9. □

Definition 8.4.11. Let $X^{\bullet,\bullet}$ be a double complex. We say that $X^{\bullet,\bullet}$ is **first quadrant** if $X^{i,j} = 0$ for all $i, j < 0$.

Proposition 8.4.12. *Let $X^{\bullet,\bullet}$ be a first quadrant double complex. Then the canonical spectral sequences E and $\mathcal{E}$ associated to X are biregular.*

Proof. Fix $n \in \mathbb{Z}$ and suppose that $p > n$. Then

$$F^p(\mathrm{Tot}(X)^n) = \bigoplus_{r \geq p} X^{r,n-r} = 0$$

since $n - r < 0$. Now suppose that $p \leq 0$. Then

$$F^p(\mathrm{Tot}(X)^n) = \bigoplus_{r \geq p} X^{r,n-r} = \bigoplus_{r \geq 0} X^{r,n-r} = \mathrm{Tot}(X)^n$$

so that the filtration F on $\mathrm{Tot}(X)$ is first quadrant. A similar argument shows that $\mathcal{F}$ is also first quadrant. The Proposition then follows by appealing to Proposition 8.3.2. □

8.5 The Grothendieck Spectral Sequence

Lemma 8.5.1 (Horseshoe Lemma)**.** *Suppose that $\mathcal{A}$ has enough injectives and let*

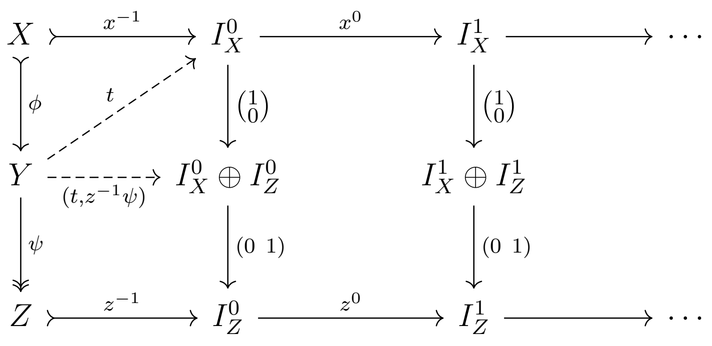
$$0 \longrightarrow X \xrightarrow{\ \phi\ } Y \xrightarrow{\ \psi\ } Z \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}$. Given injective resolutions $I_X^\bullet$ and $I_Z^\bullet$ of $X[0]$ and $Z[0]$ respectively, there exists an injective resolution $I_Y^\bullet$ of $Y[0]$ and a short exact sequence

$$0 \longrightarrow I_X^\bullet \longrightarrow I_Y^\bullet \longrightarrow I_Z^\bullet \longrightarrow 0$$

such that $I_Y^n = I_X^n \oplus I_Z^n$ for all $n \in \mathbb{N}$.

Proof. Denote by x and z the differentials of $I_X^\bullet$ and $I_Z^\bullet$ respectively. Observe that we have a diagram



Since I_X^0 is injective, there exists a morphism $t : Y \to I_X^0$ such that $x^{-1} = t \circ \phi$. We claim that $(t, z^{-1}\psi)$ is a monomorphism. To this end, suppose that $(t, z^{-1}\psi)f = 0$. Then $z^{-1}\psi f = 0$. Since z^{-1} is monic, it follows that $\psi f = 0$. Hence f factors through $\ker(\psi) = \phi$, say by g. Then $0 = tf = t\phi g = x^{-1}g$. But x^{-1} is monic whence $g = 0$ and so $f = 0$.

Now let K_0, L_0, M_0 be the cokernels of $x^{-1}, (t, z^{-1}\psi)$ and z^{-1} respectively. By the Snake Lemma, the sequence $0 \to K_0 \to L_0 \to M_0 \to 0$ is exact. We thus have the diagram

$$
\begin{array}{ccccccccc}
X & \overset{x^{-1}}{\rightarrowtail} & I_X^0 & \twoheadrightarrow & K_0 & \overset{x^0}{\rightarrowtail} & I_X^1 & \longrightarrow & \cdots \\
{\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\binom{1}{0}} & & \downarrow & & \downarrow{\scriptstyle\binom{1}{0}} & & \\
Y & \overset{(t,z^{-1}\psi)}{\rightarrowtail} & I_X^0 \oplus I_Z^0 & \twoheadrightarrow & L_0 & \dashrightarrow & I_X^1 \oplus I_Z^1 & & \\
{\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle(0\ 1)} & & \downarrow & & \downarrow{\scriptstyle(0\ 1)} & & \\
Z & \overset{z^{-1}}{\rightarrowtail} & I_Z^0 & \twoheadrightarrow & M_0 & \overset{z^0}{\rightarrowtail} & I_Z^1 & \longrightarrow & \cdots
\end{array}
$$

By the same argumentation as above, we can construct a monomorphism $L_0 \to I_X^0 \oplus I_Z^0$. Continuing in this way, we construct the claimed injective resolution of Y. □

Lemma 8.5.2. *Let $X^\bullet \in \operatorname{ob} \mathsf{Com}^+(\mathcal{A})$ be a complex. A* ***Cartan-Eilenberg resolution*** *of $X^\bullet$ is a double complex $(Y^{\bullet,\bullet}, d, \partial)$ together with a morphism $\varepsilon : X^\bullet \to Y^{\bullet,0}$ such that*

1. *$Y^{i,j} = 0$ for $j < 0$ or $i \leq 0$.*
2. *For all $i \in \mathbb{Z}$, the complex $(Y^{i,\bullet}, \varepsilon^i)$ is an injective resolution of X^i and induces injective resolutions of $Z^i(X^\bullet), B^i(X^\bullet)$ and $H^i(X^\bullet)$.*
3. *For all $i, j \in \mathbb{Z}$ the short exact sequence*

$$0 \longrightarrow \ker(d^{i,j}) \longrightarrow Y^{i,j} \longrightarrow \operatorname{im}(d^{i-1,j}) \longrightarrow 0$$

splits.

Proposition 8.5.3. *Suppose that $\mathcal{A}$ has enough injectives and let $X^\bullet \in \operatorname{ob} \mathsf{Com}^+(\mathcal{A})$ be a complex. Then $X^\bullet$ has a Cartan-Eilenberg resolution.*

Proof. Fix $i \in \mathbb{Z}$ and choose injective reslutions $I^\bullet_{B^i(X^\bullet)}$ and $I_{H^i(X^\bullet)}$ of $B^i(X^\bullet)$ and $H^iX^\bullet)$ respectively. Since we have a short exact sequence

$$0 \longrightarrow B^i(X^\bullet) \longrightarrow Z^i(X^\bullet) \longrightarrow H^i(X^\bullet) \longrightarrow 0$$

the Horseshoe Lemma provides us with an injective resolution $I_{Z^i(X^\bullet)}$ fitting into a short exact sequence

$$0 \longrightarrow I^\bullet_{B^i(X^\bullet)} \longrightarrow I^\bullet_{Z^i(X^\bullet)} \longrightarrow I^\bullet_{H^i(X^\bullet)} \longrightarrow 0$$

Similarly, the short exact sequence

$$0 \longrightarrow Z^i(X^\bullet) \longrightarrow X^i \longrightarrow B^{i+1}(X^\bullet) \longrightarrow 0$$

implies the existence of an acyclic injective resolution $I^\bullet_{X^i}$ of X^i and a short exact sequence

$$0 \longrightarrow I^\bullet_{Z^i(X^\bullet)} \longrightarrow I^\bullet_{X^i} \longrightarrow I^\bullet_{B^{i+1}(X^\bullet)} \longrightarrow 0$$

We now define a Cartan-Eilenberg resolution $(Y^{\bullet,\bullet}, d, \partial)$ of $X^\bullet$ as follows. For each i, j define $Y^{i,j} = I^j_{X^{i+1}}$. Let $d^{i,j}$ be the composition

$$I^j_{X^i} \longrightarrow I^j_{B^{i+1}(X^\bullet)} \longrightarrow I^j_{Z^{i+1}(X^\bullet)} \longrightarrow I^j_{X^{i+1}}$$

and $\partial^{i,j} = (-1)^j d^j_{I_{X^i}}$ where $d_{I_{X^i}}$ is the differential of the complex $I^\bullet_{X^i}$. It is clear that this defines a double complex satisfying the first two properties of a Cartan-Eilenberg resolution. The splitting property follows immediately from the fact that exact sequences whose first term is injective split. □

Lemma 8.5.4. *Let $X^\bullet \in \operatorname{ob} \mathsf{Com}^+(\mathcal{A})$ be a complex, $Y^{\bullet,\bullet}$ a double complex such that $Y^{i,j} = 0$ for $j < 0$ and $\varepsilon : X^\bullet \to Y^{\bullet,\bullet}$ a morphism of complexes. Suppose that $\partial \circ \varepsilon = 0$ and for each i, $Y^{\bullet,0}$ is an acyclic resolution of X^i. Then the induced morphism of complexes $\overline{\varepsilon} : X^\bullet \to \mathrm{Tot}(Y)^\bullet$ is a quasi-isomorphism.*

Proof. The fact that $\overline{\varepsilon}$ is indeed a morphism of complexes is immediate from the fact that ε is one together with the fact that $\partial \circ \varepsilon = 0$. Let $E^{p,q}_r \Rightarrow_r H^p(\mathrm{Tot}(Y))$ be the first canonical spectral sequence associated to the double complex $Y^{\bullet,\bullet}$. Then for all $q > 0$ we have

$$E^{p,q}_1 \cong H^q_\wedge(Y^{p,\bullet}) = 0$$

by acyclicity. Now,

$$H^0_\wedge(Y^{p,\bullet}) = \frac{\ker(\partial^{p,0})}{\mathrm{im}(\partial^{p,-1})} = \ker(\partial^{p,0}) \cong X^p$$

so that ε induces an isomorphism of complexes $\varepsilon : X^\bullet \to H^0_\wedge(Y^{\bullet,\bullet})$. This implies that ε induces an isomorphism on cohomology

$$\varepsilon : H^p(X^\bullet) \to E^{p,0}_2$$

But $E^{p,q}_2$ vanish for $q > 0$ and so $E^{p,q}_\infty = 0$ for $q > 0$ whence

$$H^p(\mathrm{Tot}(Y)^\bullet) = E^p \cong E^{p,0}_\infty = E^{p,0}_2 \cong H^p(X^\bullet)$$

□

Proposition 8.5.5. *Suppose that $\mathcal{A}$ has enough injectives and $F : \mathcal{A} \to \mathcal{B}$ is a left-exact functor. Let $X^\bullet \in \operatorname{ob} \mathsf{Com}^+(\mathcal{A})$ be a complex and $I^{\bullet,\bullet}$ a Cartan-Eilenberg resolution of $X^\bullet$. Then there exists a biregular spectral sequence associated to the double complex $F(I^{\bullet,\bullet})$*

$$\mathcal{E}^{p,q}_2 = \mathsf{R}^p F(H^q(X^\bullet)) \Rightarrow \mathsf{R}^{p+q} F(X^\bullet)$$

Proof. By abuse of notation, let $E^{p,q}_r$ be the second canonical spectral sequence $\mathcal{E}^{p,q}_r$ associated to the double complex $F(I^{\bullet,\bullet})$. We have

$$E^{p,q}_1 \cong H^q_>(F(I^{\bullet,p})) = F(H^q_>(I^{\bullet,p}))$$

where the second isomorphism follows from the splitting property of a Cartan-Eilenberg resolution. But $H^q_>(I^{\bullet,p})$ is an injective resolution of $H^q(X^\bullet)$ so that

$$E^{p,q}_2 \cong \mathsf{R}^q F(H^q(X^\bullet))$$

Furthermore, Lemma 8.5.4 implies that $\mathrm{Tot}(Y^\bullet)$ is an injective resolution of $X^\bullet$ so that

$$E^{p+q} = H^{p+q}(\mathrm{Tot}(F(I^{\bullet,\bullet}))) = H^{p+q}(F(\mathrm{Tot}(I^{\bullet,\bullet}))) \cong H^{p+q}(\mathsf{R}F(X^\bullet)) = \mathsf{R}^{p+q} F(X^\bullet)$$

□

Theorem 8.5.6 (Grothendieck Spectral Sequence)**.** *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be abelian categories. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and that $\mathcal{C}$ is cocomplete. Moreover, suppose that $\mathcal{B}$ has enough right $\mathfrak{R}$-objects for some admissible class $\mathfrak{R}$. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left-exact functors such that $\mathfrak{R}$ is adapted to G and F maps injective objects to $\mathfrak{R}$-objects. Then for all $X \in \operatorname{ob} \mathcal{A}$ there exists a biregular first quadrant spectral sequence*

$$E_2^{p,q} = (\mathsf{R}^p G \circ \mathsf{R}^q F)(X) \Rightarrow \mathsf{R}^{p+q}(G \circ F)(X)$$

Proof. Choose an injective resolution $I^\bullet$ of $A[0]$ and a Cartan-Eilenberg resolution $Y^{\bullet,\bullet}$ of $F(I^\bullet)$. Then Proposition 8.5.5 implies that there exists a biregular first quadrant spectral sequence

$$E_2^{p,q} = \mathsf{R}^p G(H^q(I^\bullet)) \Rightarrow \mathsf{R}^{p+q} G(F(I^\bullet))$$

On one hand, we have that

$$\mathsf{R}^p G(H^q(F(I^\bullet))) = \mathsf{R}^p G(\mathsf{R}^q F(X)))$$

and on the other hand we have

$$\begin{aligned} \mathsf{R}^{p+q} G(F(I^\bullet)) = H^{p+q}(\mathsf{R}G(\mathsf{R}F(X))) = H^{p+q}(\mathsf{R}G \circ \mathsf{R}F)(X) &\cong H^{p+q}(\mathsf{R}(G \circ F))(X) \\ &= \mathsf{R}^{p+q}(G \circ F)(X) \end{aligned}$$

where the isomorphism is provided by Proposition 5.3.7. □

Remark. The Grothendieck spectral sequence generalises and dualises in various ways. Some examples are

1. If we are simply given δ-functors $\mathsf{K}^{\mathcal{A}} \xrightarrow{F} \mathsf{K}^+(\mathcal{B}) \xrightarrow{G} \mathsf{K}^+(\mathcal{C})$ then we get a similar biregular spectral sequence in light of Remark 8.3. In this situation we would still get associated five and seven-term exact sequences but the terms would have to be appropriately shifted.

2. If we are given right-exact functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ then we can dualise the hypotheses of Theorem 8.5.6 to obtain a biregular first quadrant spectral sequence

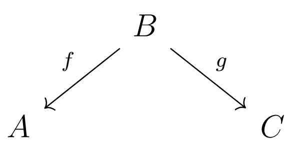
$$E^2_{p,q} = \mathsf{L}^p G(\mathsf{L}^q F(X)) \Rightarrow \mathsf{L}^{p+q}(G \circ F)(X)$$

To this spectral sequence we can also associate five and seven-term exact sequences going in the opposite direction.

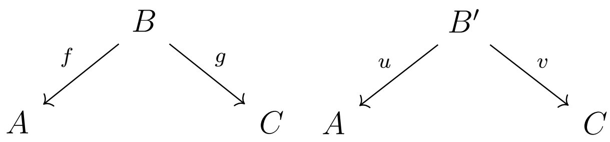
9 Appendix

9.1 Roofs

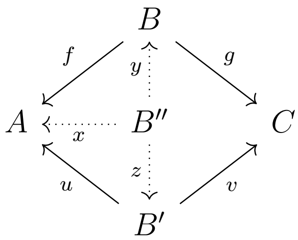
Definition 9.1.1. Let $\mathcal{C}$ be a category. A ***roof from A to C*** in $\mathcal{C}$ is a diagram of the form



It is often the case that roofs will be viewed as formal fractions in $\mathcal{C}$ and so we shall sometimes denote them as $f^{-1}g$. We say that two roofs from A to C

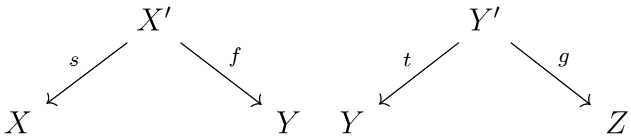


are **equivalent**, and denoted $f^{-1}g \sim u^{-1}v$, if there exists an object B'' and morphisms $x : B'' \to A, y : B'' \to B, z : B'' \to B'$ such that the diagram

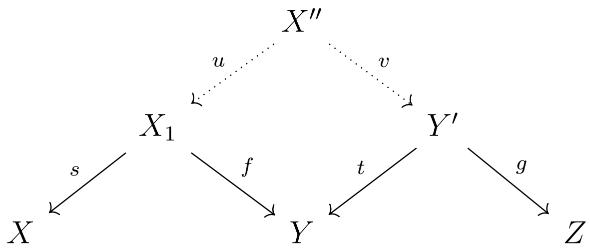


commutes.

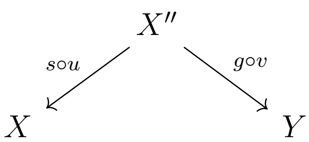
Proposition 9.1.2. *Let $\mathcal{C}$ be an additive category and S multiplicative system of morphisms of $\mathcal{C}$. Given two roofs in* $\operatorname{mor} S^{-1}\mathcal{C}$



their composition is defined using MS2 to complete the diagram

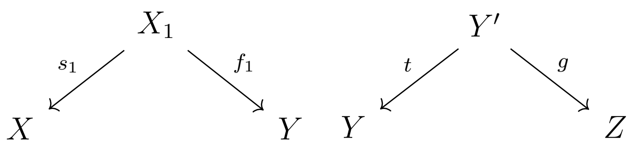


to a commutative diagram and then taking the roof



to be their composition. This construction is independent (in $S^{-1}\mathcal{C}$) of the choice of representative of $s^{-1}f$ and $t^{-1}g$ and also of the choice of commutative diagram obtained by applying MS2.

Proof. Fix two roofs



We first show that their composition is independent of the choice of representative of $s_1^{-1}f$. To this end, fix a roof

$$\begin{array}{ccc} & X_2 & \\ {}^{s_2}\swarrow & & \searrow^{f_2} \\ X & & Y \end{array}$$

that is equivalent to $s_1^{-1}f$ so that we have an object W and morphisms $x : W \to X$ in S and $y : W \to X_1$ and $z : W \to X_2$ together with a commutative diagram

$$\begin{array}{ccccc} & & X_1 & & \\ & {}^{s_1}\swarrow & \uparrow^{y} & \searrow^{f_1} & \\ X & \xleftarrow{x} & W & & Y \\ & {}_{s_2}\nwarrow & \downarrow^{z} & \nearrow_{f_2} & \\ & & X_2 & & \end{array}$$

Now form the compositions of $s_1^{-1}f_1$ and $s_2^{-1}f_2$ with $t^{-1}g$:

$$\begin{array}{ccccccccc} & & & & U_1 & & & & \\ & & & {}^{u_1}\swarrow & & \searrow^{v_1} & & & \\ & & X_1 & & & & Y' & & \\ & {}^{s_1}\swarrow & & \searrow^{f_1} & & {}^{t}\swarrow & & \searrow^{g} & \\ X & & & & Y & & & & Z \end{array} \qquad \begin{array}{ccccccccc} & & & & U_2 & & & & \\ & & & {}^{u_2}\swarrow & & \searrow^{v_2} & & & \\ & & X_1 & & & & Y' & & \\ & {}^{s_1}\swarrow & & \searrow^{f_2} & & {}^{t}\swarrow & & \searrow^{g} & \\ X & & & & Y & & & & Z \end{array}$$

Note that $u_1, u_2 \in S$. We now apply MS2 to the diagrams

$$\begin{array}{ccc} & & U_1 \\ & & \downarrow{\scriptstyle u_1} \\ W & \xrightarrow{y} & X_1 \end{array} \qquad \begin{array}{ccc} & & U_2 \\ & & \downarrow{\scriptstyle u_2} \\ W & \xrightarrow{z} & X_2 \end{array}$$

to obtain objects L_1, L_2 and morphisms $l_1 : L_1 \to W, l_2 : L_2 \to W$ in S and $m_1 : L_1 \to U_1, m_2 : L_2 \to U_2$ making the diagrams

$$\begin{array}{ccc} L_1 & \xrightarrow{m_1} & U_1 \\ \downarrow{\scriptstyle l_1} & & \downarrow{\scriptstyle u_1} \\ W & \xrightarrow{y} & X_1 \end{array} \qquad \begin{array}{ccc} L_2 & \xrightarrow{m_2} & U_2 \\ \downarrow{\scriptstyle l_2} & & \downarrow{\scriptstyle u_2} \\ W & \xrightarrow{z} & X_2 \end{array}$$

commute. We once more apply MS2 to the diagram

$$\begin{array}{ccc} & & L_1 \\ & & \downarrow{\scriptstyle l_1} \\ L_2 & \xrightarrow{l_2} & W \end{array}$$

to obtain an object R and morphisms $r_2 : R \to L_2$ in S and $r_1 : R \to L_1$ making the diagram

$$\begin{array}{ccc} R & \xrightarrow{r_1} & L_1 \\ \downarrow{\scriptstyle r_2} & & \downarrow{\scriptstyle l_1} \\ L_2 & \xrightarrow{l_2} & W \end{array}$$

commute. Observe now that

$$s_1 \circ u_1 \circ m_1 \circ r_1 = s_1 \circ y \circ l_1 \circ r_1 = s_2 \circ z \circ l_2 \circ r_2 = s_2 \circ u_2 \circ m_2 \circ r_2$$

which is in S since each of $s_2 \circ z, l_2$ and r_1 are in S. Moreover,

$$\begin{aligned} t \circ v_1 \circ m_1 \circ r_1 = f_1 \circ u_1 \circ m_1 \circ r_1 = f_1 \circ y \circ l_1 \circ r_1 = f_2 \circ z \circ l_2 \circ r_2 &= f_2 \circ u_2 \circ m_2 \circ r_2 \\ &= t \circ v_2 \circ m_2 \circ r_2 \end{aligned}$$

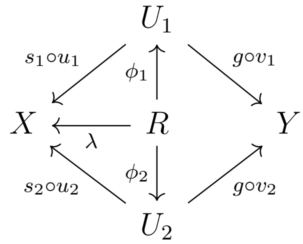
By MS3 there thus exists an object Q and a morphism $q : Q \to R$ in S such that

$$v_1 \circ m_1 \circ r_1 \circ q = v_2 \circ m_2 \circ r_2 \circ q$$

Now set $\phi_1 = m_1 \circ r_1 \circ q$ and $\phi_2 = m_2 \circ r_2 \circ q$ so that $v_1 \circ \phi_1 = v_2 \circ \phi_2$. Then

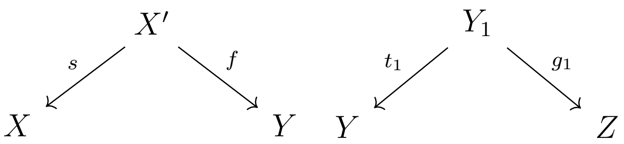
$$\lambda = s_1 \circ u_1 \circ \phi_1 = s_2 \circ u_2 \circ \phi_2$$

is in S and $g \circ v_1 \circ \phi_1 = g \circ v_2 \circ \phi_2$. We thus have a commutative diagram

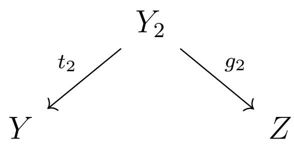


and so the two compositions are equivalent. We can now immediately conclude that the composition is also independent of the choice of commutative diagram obtained by MS2 by taking $X_2 = X_1, s_2 = s_1$ and $f_2 = f_1$ in the above proof.

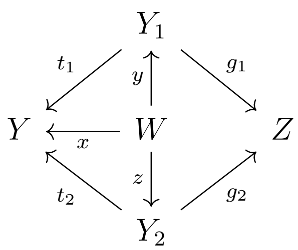
We must now show that the composition is independent of the choice of representative of $t^{-1}f$. Suppose we are given two roofs



and a roof



that is equivalent to $t_1^{-1}g_1$. That is to say, we have an object W and morphisms $x : W \to Y$ in S and $y : W \to Y_1, z : W \to Y_2$ such that there is a commutative diagram



We can apply MS2 to the diagram

$$\begin{array}{ccc} & & W \\ & & \big\downarrow{\scriptstyle t_1\circ y} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

to obtain an object U and morphisms $u : U \to X$ in S and $a : U \to W$ making the diagram

$$\begin{array}{ccc} U & \xrightarrow{\ a\ } & W \\ \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle t_1\circ y} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

commute. We thus have a commutative diagram

$$\begin{array}{ccccccc} & & & U & & & \\ & & {\scriptstyle u}\swarrow & & \searrow{\scriptstyle y\circ a} & & \\ & X & & & & Y_1 & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle f} & & {\scriptstyle t_1}\swarrow & & \searrow{\scriptstyle g_1} \\ X & & & Y & & & Z \end{array}$$

Since the composition of two roofs is independent of the choice of such a commutative diagram it follows that the roof

$$\begin{array}{ccc} & U & \\ {\scriptstyle s\circ u}\swarrow & & \searrow{\scriptstyle g_1\circ y\circ a} \\ Y & & Z \end{array}$$

represents the composition of the roofs $s^{-1}f$ and $t_1^{-1}g_1$. Note that we also have a commutative diagram

$$\begin{array}{ccccccc} & & & U & & & \\ & & {\scriptstyle u}\swarrow & & \searrow{\scriptstyle z\circ a} & & \\ & X & & & & Y_2 & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle f} & & {\scriptstyle t_2}\swarrow & & \searrow{\scriptstyle g_2} \\ X & & & Y & & & Z \end{array}$$

so that the roof

$$\begin{array}{ccc} & U & \\ {\scriptstyle s\circ u}\swarrow & & \searrow{\scriptstyle g_2\circ z\circ a} \\ Y & & Z \end{array}$$

represents the composition of the roofs $s^{-1}f$ and $t_2^{-1}g_2$. But this roof is equal to the previously constructed roof. Hence the composition of two roofs is independent of the choice of second roof. $\square$

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