Determinant Functors

Alexandre Daoud
alex.daoud@cantab.net

1 Preliminaries

1.1 Internal Groups

Definition 1.1.1. Let $\mathcal{C}$ be a category with finite products and terminal object $\mathbf{1}$. A **group internal** to $\mathcal{C}$ is given by the following data

1. An object $G \in \operatorname{ob} \mathcal{C}$.
2. A morphism $1 : \mathbf{1} \to G$ called the **unit**.
3. A morphism $(-)^{-1} : G \to G$ called the **inversion**.
4. A morphism $m : G^2 \to G$ called the **multiplication**.

such that the following **constraint** diagrams commute (possibly up to isomorphism)

$$\begin{array}{ccc} G^3 & \xrightarrow{\mathrm{id}_G \times m} & G^2 \\ \big\downarrow{\scriptstyle m\times \mathrm{id}_G} & & \big\downarrow{\scriptstyle m} \\ G^2 & \xrightarrow{m} & G \end{array} \qquad \begin{array}{ccccc} \mathbf{1}\times G & \xrightarrow{(1,\mathrm{id}_G)} & G^2 & \xleftarrow{(\mathrm{id}_G,1)} & G\times\mathbf{1} \\ & \searrow{\scriptstyle \sim} & \big\downarrow{\scriptstyle m} & {\scriptstyle \sim}\swarrow & \\ & & G & & \end{array} \qquad \begin{array}{ccccc} G & \xrightarrow{\Delta} & G^2 & \xrightarrow{((-)^{-1},\mathrm{id}_G)} & G^2 \\ \big\downarrow & & & & \big\downarrow{\scriptstyle m} \\ \mathbf{1} & & \xrightarrow{1} & & G \end{array}$$

where $\Delta : G \to G^2$ is the canonical diagonal morphism.

1.2 Complexes and Derived Categories

Let $\mathcal{A}$ be a (locally small) abelian category. For $* \in \{ \varnothing, +, -, b \}$ we denote by $\mathsf{Com}^*(\mathcal{A})$ the abelian category of unbounded, bounded from below, bounded from above, and bounded chain complexes respectively. Given $X^\bullet \in \mathsf{Com}^*(\mathcal{A})$, we denote by d_X the differential. If $f : X^\bullet \to Y^\bullet$ is a morphism of complexes, we denote by $C(f)^\bullet$ the mapping cone of f.

By $\mathsf{K}^*(\mathcal{A})$ we shall mean the homotopy category of $\mathsf{Com}^*(\mathcal{A})$ obtained by quotienting the morphism groups of $\mathcal{A}$ by the homotopy equivalence relation. This is again an additive (but not necessarily abelian) category. We say that $f : X^\bullet \to Y^\bullet$ is a quasi-isomorphism if it induces an isomorphism on cohomology and we denote by qis the collection of all quasi-isomorphisms in $\mathsf{Com}^*(\mathcal{A})$ and, by overload of notation, its image in $\mathsf{K}^*(\mathcal{A})$.

$\mathsf{K}^*(\mathcal{A})$ is naturally a triangulated category with triangles isomorphic to mapping cone diagrams of the form

$$X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow C(f)^\bullet \longrightarrow X[1]^\bullet$$

called strandard triangles. The collection of quasi-isomorphisms in $\mathsf{K}^*(\mathcal{A})$ forms a multiplicative system which is compatible with the triangulation. Localising $\mathsf{K}^*(\mathcal{A})$ at qis yields the derived category $\mathsf{D}^*(\mathcal{A})$ which is universal in the sense that any functor $F : \mathsf{K}^*(\mathcal{A}) \to \mathcal{C}$ which maps quasi-isomorphisms to isomorphisms necessarily factors through $\mathsf{D}^*(\mathcal{A})$ uniquely. $\mathsf{D}^*(\mathcal{A})$ is naturally triangulated with triangles given by all diagrams isomorphic to the image of a triangle in $\mathsf{K}^*(\mathcal{A})$ under the localisation functor $Q : \mathsf{K}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{A})$.

1.3 Exact Categories

Definition 1.3.1. Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a full additive subcategory of $\mathcal{A}$. We say that $\mathcal{B}$ is (**Quillen**) **exact** if it is closed under extensions. That is to say, given a short exact sequence

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

in $\mathcal{A}$ with A and C in $\mathcal{B}$ then B is also in $\mathcal{B}$. We shall call morphisms ϕ and ψ of $\mathcal{A}$ appearing in such exact sequences **admissible**. If E is the collection of all such exact sequences in $\mathcal{A}$ then we shall sometimes refer to $\mathcal{B}$ as the pair $(\mathcal{B}, E)$ to make explicit the distinguished collection of exact sequences.

Definition 1.3.2. Let $\mathcal{B}$ be an exact category which is a full subcategory of an abelian category $\mathcal{A}$. We define $\mathsf{Com}^*(\mathcal{B})$ (resp. $\mathsf{K}^*(\mathcal{B})$, $\mathsf{D}^*(\mathcal{B})$) for $* \in \{\varnothing, +, -, b\}$ to be the full subcategory of $\mathsf{Com}^*(\mathcal{A})$ (resp. $\mathsf{K}^*(\mathcal{A})$, $\mathsf{D}^*(\mathcal{A})$) consisting of complexes whose every component is isomorphic to an object of $\mathcal{B}$.

2 Picard Categories

2.1 Definitions

Definition 2.1.1. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a **groupoid** if every morphism in $\mathcal{C}$ is invertible. We denote by Grpd the category of groupoids and functors between them.

Remark. Note that Grpd has all finite products as well as the trivial category as its terminal object.

Definition 2.1.2. We define a **Picard** category $\mathcal{P}$ to be a group internal to Grpd. We denote by $\otimes$ the multiplication bifunctor implicit in the internal group structure of $\mathcal{P}$.

Proposition 2.1.3. *Let $\mathcal{P}$ be a Picard category. Then $\mathcal{P}$ is a monoidal category with tensor functor $\otimes$.*

Proof. This proof is straight-forward and follows from the relevant definitions. □

Definition 2.1.4. Let $\mathcal{P}$ be a Picard category. We say that $\mathcal{P}$ is **commutative** if it is symmetric as a monoidal category.

2.2 The Picard Category of Graded Lines

Definition 2.2.1. Let R be a commutative ring. We define the category of **graded lines** over R, denoted $\mathsf{line}_R^{\mathbb{Z}}$, to be the one given by the following data

1. The objects are pairs (L, α) where L is an invertible R-module and $\alpha : \mathrm{Spec}(R) \to \mathbb{Z}$ is a locally constant function.

2. The morphisms $f : (L, \alpha) \to (M, \beta)$ are isomorphisms of R-modules $h : L \to M$ such that whenever $\mathfrak{p} \in \mathrm{Spec}(R)$ and $\alpha(\mathfrak{p}) \neq \beta(\mathfrak{p})$ then $f_{\mathfrak{p}} : L_{\mathfrak{p}} \to M_{\mathfrak{p}}$ is trivial.

Proposition 2.2.2. *Let R be a commutative ring. Then $\mathsf{line}_R^{\mathbb{Z}}$ admits the structure of a commutative Picard category by defining*

1. *the multiplication to be*

$$(L, \alpha) \otimes (M, \beta) = (L \otimes_R M, \alpha + \beta),$$

2. *the unit object to be the trivial graded line $(R, 0)$,*

3. *the inverse of a graded line (L, α) to be $(L^{-1}, -\alpha)$,*

4. *the commutativity constraint to be given by the isomorphisms*

$$\psi_{(L,\alpha),(M,\beta)}(l \otimes m) = (-1)^{\alpha(\mathfrak{p})\beta(\mathfrak{p})}(m \otimes l)$$

whenever $l \in L_{\mathfrak{p}}$ and $m \in M_{\mathfrak{p}}$.

Proof. This proof is straight-forward and follows from the relevant definitions. □

Remark. Note that if X is any ringed space then we can make a similar definition of the category of graded line bundles $\mathsf{line}_X^{\mathbb{Z}}$ consisting of pairs (L, α) where L is an invertible $\mathcal{O}_X$-module and $\alpha : X \to \mathbb{Z}$ is a locally constant function. In the particular case where X is the affine scheme $\mathrm{Spec}(R)$, this reduces to the previous definition. This definition also provides motivation for the terminology of a Picard category since the isomorphism classes of invertible $\mathcal{O}_X$-modules over a ringed space X form a group typically called the Picard group of X.

3 Abstract Determinant Functors

3.1 Definitions and Basic Properties

Definition 3.1.1. Let $\mathcal{A}$ be an exact category and $w \subseteq \mathrm{mor}\,\mathcal{A}$ a collection of morphisms. We say that w is a **SQ-class** if it satisfies the following properties

1. Every isomorphism is in w.

2. If any two of f, g and $g \circ f$ are in w then so is the third.

3. Given morphisms of short exact sequences α, β and γ such that any two of them are in w then so is the third.

We denote by $\mathcal{A}_w$ the subcategory of $\mathcal{A}$ whose morphisms are w. We will often just call $\mathcal{A}_w$ exact and assume that w is given implicitly.

Example 3.1.2. Given any exact category $\mathcal{A}$, the collection of isomorphisms iso in $\mathrm{mor}\,\mathcal{A}$ is an SQ-class. Moreover, the collection of quasi-isomorphisms qis in $\mathrm{mor}\,\mathsf{Com}^b(\mathcal{A})$ is also an SQ-class.

Definition 3.1.3. Let $\mathcal{A}_w$ be an exact category. A **determinant functor** on $\mathcal{A}_w$ is a choice of commutative Picard category $\mathcal{P}$ and a functor $\mathsf{d} : \mathcal{A}_w \to \mathcal{P}$ together with the data

DF1 For every short exact sequence Σ

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$, an isomorphism

$$\mathsf{d}(\Sigma) : \mathsf{d}(B) \xrightarrow{\sim} \mathsf{d}(A) \otimes \mathsf{d}(C)$$

which is functorial in w-morphisms of short exact sequences.

DF2 An isomorphism $\zeta(0) : \mathsf{d}(0) \xrightarrow{\sim} \mathbf{1}_{\mathcal{P}}$ where $\mathbf{1}_{\mathcal{P}}$ is the unit object of $\mathcal{P}$.

subject to the following axioms

DF3 Let $\phi : A \to B$ be an isomorphism in $\mathcal{A}_w$ giving rise to the short exact sequences

$$\Sigma' : 0 \longrightarrow 0 \longrightarrow A \xrightarrow{\phi} B \longrightarrow 0$$

$$\Sigma' : 0 \longrightarrow A \xrightarrow{\phi} B \longrightarrow 0 \longrightarrow 0$$

Then $\mathsf{d}(\phi)$ and $\mathsf{d}(\phi^{-1})$ are the compositions

$$\mathsf{d}(A) \xrightarrow{\mathsf{d}(\Sigma)} \mathsf{d}(0) \otimes \mathsf{d}(B) \xrightarrow{\zeta(0)\otimes \mathrm{id}_{\mathsf{d}(B)}} \mathsf{d}(B)$$

$$\mathsf{d}(B) \xrightarrow{\mathsf{d}(\Sigma')} \mathsf{d}(A) \otimes \mathsf{d}(0) \xrightarrow{\mathrm{id}_{\mathsf{d}(A)}\otimes \zeta(0)} \mathsf{d}(A)$$

respectively.

DF4 Given admissible subobjects $0 \subseteq A \subseteq B \subseteq C$ of an object C in $\mathcal{A}_w$, the diagram

$$\begin{array}{ccc} \mathsf{d}(C) & \longrightarrow & \mathsf{d}(A) \otimes \mathsf{d}(C/A) \\ \downarrow & & \downarrow \\ \mathsf{d}(B) \otimes \mathsf{d}(C/B) & \longrightarrow & \mathsf{d}(A) \otimes \mathsf{d}(B/A) \otimes \mathsf{d}(C/B) \end{array}$$

commutes.

Proposition 3.1.4. *Let $\mathcal{A}$ be an exact category and $\mathsf{d} : \mathcal{A}_w \to \mathcal{P}$ a determinant functor. Given $A, B \in \operatorname{ob} \mathcal{A}_w$, there is an isomorphism*

$$\mathsf{d}(A \oplus B) \cong \mathsf{d}(A) \otimes \mathsf{d}(B)$$

Proof. A and B fit into a canonical exact sequence

$$\Sigma : 0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0$$

which yields the isomorphism $\mathsf{d}(\Sigma) : \mathsf{d}(A \oplus B) \cong \mathsf{d}(A) \otimes \mathsf{d}(B)$. $\square$

3.2 Determinant Functors on Chain Complexes

Proposition 3.2.1. *Let* $\mathsf{d} : \mathcal{A}_{\text{iso}} \to \mathcal{P}$ *be a determinant functor. Then* d *induces a determinant functor*

$$\mathsf{d} : \mathsf{Com}^b(\mathcal{A}_{\text{iso}})_{\text{qis}} \to \mathcal{P}$$

given by setting

$$\mathsf{d}(X^\bullet) = \bigotimes_{i\in\mathbb{Z}} \mathsf{d}(X^i)^{(-1)^i}$$

Proof. We only define d on morphisms. The rest of the proof can be found in [KM76]. To this end, first suppose that $X^\bullet \in \operatorname{ob}\mathsf{Com}^b(\mathcal{A}_{\text{iso}})$ is acyclic. Let I^i be the image of d_X^i. Then we have an exact sequence

$$0 \longrightarrow I^{i-1} \longrightarrow X^i \longrightarrow I^i \longrightarrow 0$$

for each $i \in \mathbb{Z}$. Hence for each $i \in \mathbb{Z}$ we have an isomorphism

$$\mathsf{d}(X^i) \cong \mathsf{d}(I^{i-1}) \otimes \mathsf{d}(I^i)$$

Taking the alternating tensor product over all $i \in \mathbb{Z}$ of this isomorphism yields a canonical isomorphism $\mathsf{d}(X^\bullet) \cong \mathbf{1}_{\mathcal{P}}$. Now let $f : X^\bullet \to Y^\bullet$ be a quasi-isomorphism. Then the mapping cone $C(f)^\bullet$ of f fits into an exact sequence

$$0 \longrightarrow Y^\bullet \longrightarrow C(f)^\bullet \longrightarrow X[1]^\bullet \longrightarrow 0$$

which induces an isomorphism $\mathsf{d}(C(f)^\bullet) \cong \mathsf{d}(Y^\bullet)\otimes\mathsf{d}(X[1]^\bullet)$. But a morphism of complexes is a quasi-isomorphism if and only if its mapping cone is acyclic so that we have isomorphisms

$$\mathbf{1}_{\mathcal{P}} \cong \mathsf{d}(C(f)^\bullet) \cong \mathsf{d}(Y^\bullet) \otimes \mathsf{d}(X[1]^\bullet) \cong \mathsf{d}(Y^\bullet) \otimes \mathsf{d}(X^\bullet)^{-1}$$

whence an isomorphism $\mathsf{d}(X^\bullet) \cong \mathsf{d}(Y^\bullet)$. We then define the value of $\mathsf{d}(f)$ to be this isomorphism. □

Proposition 3.2.2. *Let* $\mathcal{A}$ *be an exact category and* $\mathsf{d} : \mathcal{A}_{\text{iso}} \to \mathcal{P}$ *a determinant functor. Let* $X^\bullet \in \mathsf{Com}^b(\mathcal{A})$ *be a complex with greatest lower bound* $n \in \mathbb{Z}$*. Suppose that* $X^\bullet$ *admits a filtration* $F^\bullet(X^\bullet)$ *such that* $F^p(X^\bullet) \in \mathsf{Com}^b(\mathcal{A})$ *and for* $p \leq n$ *we have* $F^p(X^i) = X^i$*. Then there is a canonical isomorphism*

$$\mathsf{d}(X^\bullet) \cong \bigotimes_{i\in\mathbb{Z}} \mathsf{d}(\operatorname{gr}_i(X^\bullet))$$

where $\operatorname{gr}_i(X^\bullet) = F^i(X^\bullet)/F^{i+1}(X^\bullet)$ *is the* i^{th} *graded part of* $X^\bullet$*.*

Proof. Without loss of generality, we may assume that $n = 0$ so that F is a first-quadrant filtration. First observe that, since $F^0(X^\bullet) = X^\bullet$, we have a short exact sequence

$$0 \longrightarrow F^1(X^\bullet) \longrightarrow X^\bullet \longrightarrow \operatorname{gr}_0(X^\bullet) \longrightarrow 0$$

yielding an isomorphism $\mathsf{d}(X^\bullet) \cong \mathsf{d}(F^1(X^\bullet)) \otimes \mathsf{d}(\operatorname{gr}_0(X^\bullet))$. Similarly, the short exact sequence

$$0 \longrightarrow F^2(X^\bullet) \longrightarrow F^1(X^\bullet) \longrightarrow \operatorname{gr}_1(X^\bullet) \longrightarrow 0$$

yields an isomorphism $\mathsf{d}(F^1(X^\bullet)) \cong \mathsf{d}(F^2(X^\bullet)) \otimes \mathsf{d}(\mathrm{gr}_1(X^\bullet))$. This combines with the previous isomorphism to provide an isomorphism

$$\mathsf{d}(X^\bullet) \cong \mathsf{d}(F^2(X^\bullet)) \otimes \mathrm{gr}_0(X^\bullet) \otimes \mathrm{gr}_1(X^\bullet)$$

Continuing in this fashion, we construct the desired isomorphism of the Proposition. □

Proposition 3.2.3. *Let $\mathcal{A}$ be an exact category and $\mathsf{d} : \mathcal{A}_{\mathrm{iso}} \to \mathcal{P}$ a determinant functor. Given a complex $X^\bullet \in \mathsf{Com}^b(\mathcal{A})$, suppose that $H^i(X^\bullet) \in \mathrm{ob}\,\mathcal{A}$ for all $i \in \mathbb{Z}$. Then there is a canonical isomorphism*

$$\mathsf{d}(X^\bullet) \cong \bigotimes_{i\in\mathbb{Z}} \mathsf{d}(H^i(X^\bullet))^{(-1)^i}$$

that is functorial in quasi-isomorphisms.

Proof. Denote $Z^i(X^\bullet) = \ker(d^i_X)$ and $B^i(X^\bullet) = \mathrm{im}(d^{i-1}_X)$ so that $H^i(X^\bullet) = Z^i(X^\bullet)/B^i(X^\bullet)$. Note that the associativity axiom of d provides us with an isomorphism

$$\mathsf{d}(X^i) \cong \mathsf{d}(B^i(X^\bullet)) \otimes \mathsf{d}(H^i(X^\bullet)) \otimes \mathsf{d}(X^i/Z^i(X^\bullet))$$

Moreover, we have a short exact sequence

$$0 \longrightarrow Z^i(X^\bullet) \longrightarrow X^i \longrightarrow B^{i+1}(X^\bullet) \longrightarrow 0$$

so that $X^i/Z^i(X^\bullet) \cong B^{i+1}(X^\bullet)$. This induces an isomorphism

$$\mathsf{d}(X^i) \cong \mathsf{d}(B^i(X^\bullet)) \otimes \mathsf{d}(H^i(X^\bullet)) \otimes \mathsf{d}(B^{i+1}(X^\bullet))$$

Passing to the alternating tensor product over $i \in \mathbb{Z}$, it is then clear that

$$\mathsf{d}(X^\bullet) \cong \bigotimes_{i\in\mathbb{Z}} \mathsf{d}(H^i(X^\bullet))^{(-1)^i}$$

We omit the proof of functoriality. □

3.3 Determinant Functors on Derived Categories

Throughout this section, let $\mathcal{A} = \mathcal{A}_{\mathrm{iso}}$ be an exact category and $\mathsf{d} : \mathcal{A} \to \mathcal{P}$ a determinant functor.

We first recall the mapping cylinder construction. Let $f : X^\bullet \to Y^\bullet$ be a morphism of complexes in $\mathcal{A}$. Then the mapping cylinder is the complex given by the data

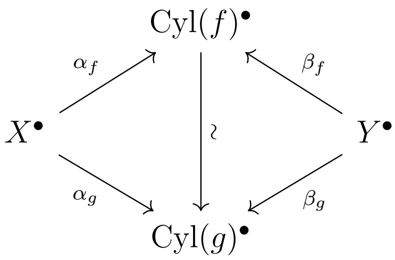
$$\mathrm{Cyl}(f)^\bullet = X[1]^\bullet \oplus X^\bullet \oplus Y^\bullet$$
$$d^i_{\mathrm{Cyl}(f)} = \begin{pmatrix} -d^{i+1}_X & 0 & 0 \\ -\mathrm{id}^{i+1}_{X^\bullet} & d^i_X & 0 \\ f^{i+1} & 0 & d^i_Y \end{pmatrix}$$

Observe that there are canonical morphisms

$$X^\bullet \xrightarrow{\alpha_f} \mathrm{Cyl}(f)^\bullet \underset{\beta_f}{\overset{\beta'_f}{\rightleftarrows}} Y^\bullet$$

which are quasi-isomorphisms and satisfy the relations $f = \beta'_f \circ \alpha_f$ and $\beta'_f \circ \beta_f = \mathrm{id}_{Y^\bullet}$.

Lemma 3.3.1. *Let $f, g : X^\bullet \rightrightarrows Y^\bullet$ be a parallel pair of morphisms of complexes in $\mathcal{A}$, If f is homotopic to g then there exists an isomorphism $\mathrm{Cyl}(f)^\bullet \cong \mathrm{Cyl}(g)^\bullet$ such that the diagram*



commutes.

Proof. Suppose that f is homotopic to g via the homotopy operator k. Then it is easy to verify that the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 0 \end{pmatrix}$$

is an isomorphism $\mathrm{Cyl}(f)^\bullet \xrightarrow{\sim} \mathrm{Cyl}(g)^\bullet$. In fact, the proof of this statement is almost identical to the proof of the same statement for the mapping cone (see [HA, Proposition 4.3.5]). □

Proposition 3.3.2. $\mathsf{d} : \mathsf{Com}^b(\mathcal{A})_{\mathrm{qis}} \to \mathcal{P}$ *descends to a functor*

$$\mathsf{d} : \mathsf{K}^b(\mathcal{A})_{\mathrm{qis}} \to \mathcal{P}$$

Proof. We need to show that d is constant on homotopy classes of morphisms. To this end, suppose that $f, g : X^\bullet \to Y^\bullet$ is a parallel pair of morphisms such that f is homotopic to g. By the mapping cylinder construction, we have that

$$\begin{aligned} \mathsf{d}(f) = \mathsf{d}(\beta_f) \circ \mathsf{d}(\alpha_f) = \mathsf{d}(\beta'_f) \circ \mathsf{d}(\beta_f) \circ \mathsf{d}(\beta_f)^{-1} \circ \mathsf{d}(\alpha_f) &= \mathsf{d}(\beta'_f \circ \beta_f) \circ \mathsf{d}(\beta_f)^{-1} \circ \mathsf{d}(\alpha_f) \\ &= \mathsf{d}(\mathrm{id}_{Y^\bullet}) \circ \mathsf{d}(\beta_f)^{-1} \circ \mathsf{d}(\alpha_f) \\ &= \mathsf{d}(\beta_f)^{-1} \circ \mathsf{d}(\alpha_f) \end{aligned}$$

and similarly for $\mathsf{d}(g)$. But by Lemma 3.3.1,

$$\mathsf{d}(\beta_f)^{-1} \circ \mathsf{d}(\alpha_f) = \mathsf{d}(\beta_g)^{-1} \circ \mathsf{d}(\alpha_g)$$

so that $\mathsf{d}(f) = \mathsf{d}(g)$ as claimed. □

Recall that the triangles of $\mathsf{D}^b(\mathcal{A})$ are diagrams that are isomorphic to a so-called standard triangle

$$X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow C(f)^\bullet \longrightarrow X[1]^\bullet$$

Moreover, to each short exact sequence

$$0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0$$

in $\mathsf{Com}^b(\mathcal{A})$ is associated functorially a triangle

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow X[1]^\bullet$$

in $\mathsf{D}^b(\mathcal{A})$ via the mapping cylinder construction (see HA, Proposition 4.6.3). We shall call such a triangle a **true triangle**. By a **true nine-term diagram** we shall mean a diagram of the form

$$\begin{array}{ccccc} X_1^\bullet & \xrightarrow{u_1} & Y_1^\bullet & \xrightarrow{v_1} & Z_1^\bullet \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X_2^\bullet & \xrightarrow{u_2} & Y_2^\bullet & \xrightarrow{v_2} & Z_2^\bullet \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle h'} \\ X_3^\bullet & \xrightarrow{u_3} & Y_3^\bullet & \xrightarrow{v_3} & Z_3^\bullet \end{array}$$

in which each row and column is a true triangle.

Theorem 3.3.3. $\mathsf{d} : \mathsf{K}^b(\mathcal{A})_{\mathrm{qis}} \to \mathcal{P}$ *extends to a unique functor*

$$\mathsf{d} : \mathsf{D}^b(\mathcal{A})_{\mathrm{iso}} \to \mathcal{P}$$

such that

1. *For every triangle*

$$\Delta : X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \longrightarrow X[1]^\bullet$$

   there exists an isomorphism

$$\mathsf{d}(\Delta) : \mathsf{d}(Y^\bullet) \xrightarrow{\sim} \mathsf{d}(X^\bullet) \otimes \mathsf{d}(Z^\bullet)$$

   which is functorial in isomorphisms of triangles $\phi : \Delta \to \Delta'$ *in the following cases*

   - Δ *and* Δ' *are true triangles.*
   - *Each complex appearing in* Δ *and* Δ' *have the property that their cohomology objects are in* $\mathcal{A}$*.*

2. *If* Δ *is a true triangle and* u *(resp.* v*) is an isomorphism then* $\mathsf{d}(u) = \mathsf{d}(\Delta)^{-1}$ *(resp.* $\mathsf{d}(v) = \mathsf{d}(\Delta)$*).*

3. *For any true nine-term diagram in* $\mathsf{D}^b(\mathcal{A})$ *we have a commutative diagram*

$$\begin{array}{ccc} \mathsf{d}(Y_2^\bullet) & \longrightarrow & \mathsf{d}(X_1^\bullet) \otimes \mathsf{d}(Z_1^\bullet) \\ \downarrow & & \downarrow \\ \mathsf{d}(Y_1^\bullet) \otimes \mathsf{d}(Y_3^\bullet) & \longrightarrow & \mathsf{d}(X_1^\bullet) \otimes \mathsf{d}(Z_1^\bullet) \otimes \mathsf{d}(X_3^\bullet) \otimes \mathsf{d}(Z_3^\bullet) \end{array}$$

   commutes.

Proof.

<u>Part 1:</u> Since $\mathsf{d} : \mathsf{K}^b(\mathcal{A}) \to \mathcal{P}$ sends quasi-isomorphisms to isomorphisms, the universal property of the localisation functor $Q : \mathsf{K}^b(\mathcal{A}) \to \mathsf{D}^b(\mathcal{A})$ implies that there is a unique functor $\mathsf{d} : \mathsf{D}^b(\mathcal{A}) \to \mathcal{P}$ such that the diagram

$$\begin{array}{ccc} \mathsf{K}^b(\mathcal{A}) & \xrightarrow{Q} & \mathsf{D}^b(\mathcal{A}) \\ & \searrow{\scriptstyle \mathsf{d}} & \vdots{\scriptstyle \mathsf{d}} \\ & & \mathcal{P} \end{array}$$

commutes. This extension is defined on morphisms as follows. Fix an isomorphism $f : X^\bullet \to Y^\bullet$. Then f is represented by a roof $s^{-1}g$ where both s^{-1} and g are quasi-isomorphisms in $\mathsf{K}^b(\mathcal{A})$. Then $\mathsf{d}(f) = \mathsf{d}(g) \circ \mathsf{d}(s)^{-1}$.

Now fix a triangle Δ as in the Theorem and suppose that it does not fall within one of the two distinguished cases. Then Δ is isomorphic to a standard triangle

$$\Delta' : A^\bullet \xrightarrow{f} B^\bullet \longrightarrow C(f)^\bullet \longrightarrow X[1]^\bullet$$

for some $f : A^\bullet \to B^\bullet$ in $\mathsf{D}^b(\mathcal{A})$, say via $\phi : \Delta \to \Delta'$. Since the standard triangles in $\mathsf{D}^b(\mathcal{A})$ are just the images under the localisation of standard triangles in $\mathsf{K}^b(\mathcal{A})$, we may assume that f is a morphism in $\mathsf{K}^b(\mathcal{A})$. Now note that the mapping cone $C(f)$ fits into the short exact sequence

$$0 \longrightarrow B^\bullet \longrightarrow C(f) \longrightarrow A[1]^\bullet \longrightarrow 0$$

so that d on $\mathsf{K}^b(\mathcal{A})$ yields isomorphisms

$$\mathsf{d}(C(f)) \cong \mathsf{d}(B^\bullet) \otimes \mathsf{d}(A[1]^\bullet) \cong \mathsf{d}(B^\bullet) \otimes \mathsf{d}(A^\bullet)^{-1}$$

We thus have an isomorphism $\mathsf{d}(B)^\bullet \cong \mathsf{d}(A^\bullet) \otimes \mathsf{d}(C(f)^\bullet)$. Composing this isomorphism with the isomorphism $\mathsf{d}(\phi)$ we get an isomorphism $\mathsf{d}(Y^\bullet) \cong \mathsf{d}(X^\bullet) \otimes \mathsf{d}(Z^\bullet)$. This defines the desired isomorphism $\mathsf{d}(\Delta)$.

Now suppose that Δ is a true triangle. Then there exists a short exact sequence

$$0 \longrightarrow X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \longrightarrow 0$$

in $\mathsf{Com}^b(\mathcal{A})$ yielding an isomorphism $\mathsf{d}(Y^\bullet) \cong \mathsf{d}(X^\bullet) \otimes \mathsf{d}(Z^\bullet)$. This gives the desired isomorphism $\mathsf{d}(\Delta)$ which is functorial with respect to isomorphisms of true triangles since the assignment to each true triangle its corresponding short exact sequence is a functorial one.

Assume now that Δ has the property that every complex appearing in it has cohomology objects in $\mathcal{A}$. Then by Proposition 3.2.3, we have an isomorphism

$$\mathsf{d}(Y^\bullet) \cong \bigotimes_{i \in \mathbb{Z}} \mathsf{d}(H^i(Y^\bullet))^{(-1)^i}$$

which is functorial in isomorphisms (and similarly for $X^\bullet$ and $Z^\bullet$). But note that the long exact cohomology sequence $H(\Delta)^\bullet$ associated to Δ is an acyclic complex in $\mathsf{D}^b(\mathcal{A})$ and so

$$\begin{aligned} \mathbf{1}_{\mathcal{P}} \cong \mathsf{d}(H(\Delta)^\bullet) &= \bigotimes_{i \in \mathbb{Z}} \mathsf{d}(H(\Delta)^i)^{(-1)^i} \\ &\cong \left(\bigotimes_{i \in \mathbb{Z}} \mathsf{d}(H^i(X^\bullet))^{(-1)^i} \right) \otimes \left(\bigotimes_{i \in \mathbb{Z}} \mathsf{d}(H^i(Y^\bullet))^{(-1)^{i+1}} \right) \otimes \left(\bigotimes_{i \in \mathbb{Z}} \mathsf{d}(H^i(Z^\bullet))^{(-1)^i} \right) \end{aligned}$$

This yields isomorphisms

$$\begin{aligned} \mathsf{d}(Y^\bullet) \cong \bigotimes_{i \in \mathbb{Z}} \mathsf{d}(H^i(Y^\bullet))^{(-1)^i} &\cong \left(\bigotimes_{i \in \mathbb{Z}} \mathsf{d}(H^i(X^\bullet))^{(-1)^i} \right) \otimes \left(\bigotimes_{i \in \mathbb{Z}} \mathsf{d}(H^i(Z^\bullet))^{(-1)^i} \right) \\ &\cong \mathsf{d}(X^\bullet) \otimes \mathsf{d}(Z^\bullet) \end{aligned}$$

which are all functorial in isomorphisms.

Part 2: Suppose that Δ is a true triangle. By the definition of the triangulation structure on $\mathsf{D}^b(\mathcal{A})$, Δ is isomorphic to a standard triangle

$$\Delta' : A^\bullet \xrightarrow{f} B^\bullet \longrightarrow C(f) \longrightarrow A[1]^\bullet$$

with f an isomorphism. But then $C(f)$ is acyclic whence so is $Z^\bullet$. So we may assume that $Z^\bullet = 0$ and u is an honest isomorphism in $\mathsf{Com}^b(\mathcal{A})$. Then $\mathsf{d}(\Delta)$ is the isomorphism $\mathsf{d}(Y^\bullet) \cong \mathsf{d}(X^\bullet)$ coming from the exact sequence

$$0 \longrightarrow X^\bullet \xrightarrow{\ u\ } Y^\bullet \longrightarrow 0 \longrightarrow 0$$

which coincides with $\mathsf{d}(u)$ by DF3 for d on $\mathcal{A}$.

Part 3: Omitted.

$\square$

References

[BF01] D. Burns and M. Flach. “Tamagawa Numbers for Motives with (Noncommutative) Coefficients”. In: *Documenta Mathematica* 6.3 (2001), pp. 501–570. ISSN: 00029327, 10806377. URL: `http://www.jstor.org/stable/25099184`.

[Bur11] David Burns. “An Introduction to the Equivariant Tamagawa Number Conjecture: the Relation to Stark’s Conjecture”. In: *Arithmetic of L-functions*. Vol. 18. IAS / Park City Mathematics. American Mathematical Society, 2011, pp. 125–152.

[Dao] Alexandre Daoud. *Homological Algebra*. HA. URL: `http://www.p-adic.com/Homological%20Algebra.pdf`.

[FP91] J.-M. Fontaine and B. Perrin-Riou. “Autor des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs des fonctions L”. In: *Motives*. Vol. 55.1. Proceedings of Symposia in Pure Mathematics. American Mathematical Society, 1991, pp. 599–706.

[Kak12] M. Kakde. “From the classical to the noncommutative Iwasawa theory (for totally real number fields)”. In: *Non-abelian Fundamental Groups and Iwasawa Theory*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2012, pp. 107–131.

[KM76] Finn Knudsen and David Mumford. “The projectivity of the moduli space of stable curves. I: Preliminaries on "det" and "Div".” In: *Mathematica Scandinavica* 39.0 (1976), pp. 19–55. ISSN: 1903-1807. DOI: `10.7146/math.scand.a-11642`. URL: `http://www.mscand.dk/article/view/11642`.

[Mur] Fernando Muro. *On determinants (as functors)*. URL: `http://personal.us.es/fmuro/files/slides/pontevedra.pdf`.

[Ven07] O. Venjakob. “From the Birch and Swinnerton-Dyer Conjecture to non-commutative Iwasawa theory via the Equivariant Tamagawa Number Conjecture - a survey”. In: *L-functions and Galois Representations*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2007, pp. 333–380.

[Zäh12] Yasin Zähringer. “Tamagawa Number Conjecture for Semi-Abelian Varieties”. MA thesis. Ruprecht-Karls-Universität Heidelberg, Sept. 2012. URL: `https://www.mathi.uni-heidelberg.de/~otmar/diplom/zaehringer.pdf`.